On the optimal passive and semi-active damping control in free vibration of a Single-degree-of-freedom system

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I. INTRODUCTION

The semi-active damping affects a system by the use of a controllable device that alters its damping. This method has attracted scientists and engineers working in vibration control. Some of the semi-active damping devices include: variable-orifice fluid dampers [1], controllable friction devices [2], variable-stiffness devices [3], semi-active tuned mass dampers [4, 5, 6], semi-active tuned liquid dampers [7], controllable fluid dampers [8], controllable-inertia flywheels based on moving masses [9], MR elastomers [10], vacuum-packed semi-active dampers [11].

While there are many semi-active controllers have been proposed, due to the intrinsic nonlinearities and physical constraints, finding the closed form solutions of the optimal control problem is very challenging.

Under harmonic excitation, works have been conducted to optimize some classes of switching controller [12-13]. However, under free vibration, the optimization problem met difficulties.

This paper presents the closed form of minimum of integral of mechanical energy of Single-degree-of-freedom (SDOF) passive and semi-active damping system.

II. INTEGRAL OF MECHANICAL ENERGY FOR A SDOF SYSTEM

The model of the semi-active damping controlled SDOF system are shown in Fig.1.

Abstract— This paper considers the minimization of the normalized integrals of mechanical energy of passive a semi-active damping Single-degree-of-freedom (SDOF) systems. The analytical values of the optimal performance index are 0.707 and 0.643, respectively for the passive, and semi-active damping controlled systems. The optimal passive damping ratio is 0.707. The optimal semi-active damping should switch the system from the undamped to the critically damped at the time of 1/8 of the undamped natural period.

Keywords— Analytical optimization, bi-linear state space system, quadratic integral.

Fig. 1. Semi-active damping SDOF

In Fig. 1, the mass \( m \) is supported by the spring \( k \) and the semi-active damper \( c \) and is free vibrated with the displacement \( z \). For demonstration, the initial displacement condition is considered. The initial velocity problem should be studied in the future. The initial condition is taken as:

\[
z(0) = z_0, \quad \dot{z}(0) = 0 \tag{1}
\]

The free vibrations of the systems in Fig.1 are driven by the following classical equations:

\[
\begin{aligned}
\ddot{z}(t) + 2\zeta_i\omega_n\dot{z}(t) + \omega_n^2 z(t) &= 0; \quad 0 \leq t \leq t_1 \\
\ddot{z}(t) + 2\zeta_i\omega_n\dot{z}(t) + \omega_n^2 z(t) &= 0; \quad t_1 \leq t
\end{aligned}
\tag{2}
\]

where: \( \omega_n = \sqrt{k/m} \) is the undamped nominal natural circular frequency, \( \zeta_i \) (\( i = 1,2 \)) are the damping ratios of the bi-state damping, \( t_1 \) is the switching time, which is the key parameters affecting the performance

The performance index is considered as:

\[
J = \frac{1}{\omega_n} \int_{0}^{\infty} \left( \omega_n^2 z^2 + \dot{z}^2 \right) \frac{dt}{2z_0^2} \tag{3}
\]
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In fact, the expression under the integral is the normalized mechanical energy. It is noted that the performance index (3) is normalized to be nondimensional. Denote the state vector as:

\[
x = \frac{1}{z_0} \begin{bmatrix} z \ x \ z \
\end{bmatrix}^T
\]

(4)

with the initial condition

\[
x_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T
\]

(5)

The performance index (3) is written in the quadratic integral as:

\[
J = \int_0^t x^T Q x dt + \int_0^t x^T Q x dt
\]

(6)

where the weighting matrix is:

\[
Q = \frac{\omega_n}{2} \begin{bmatrix} 1 & 0 \\
0 & 0 \end{bmatrix}
\]

(7)

The equation (2) is written in the state space form:

\[
\begin{align*}
\dot{x} &= A_1 x, \quad 0 < t < t_1 \\
\dot{x} &= A_2 x, \quad t_1 < t \\
x(0) &= x_0
\end{align*}
\]

(8)

where the state vector is (4) and the system matrices are:

\[
A_1 = \omega_n \begin{bmatrix} 0 & 1 \\
-1 & -2\zeta_1 \end{bmatrix}
\]

(9)

\[
A_2 = \omega_n \begin{bmatrix} 0 & 1 \\
-1 & -2\zeta_2 \end{bmatrix}
\]

The solution of the bi-linear system (8) can be found as:

\[
x(t) = e^{A_1 t} x_0 + \sum_{j=0}^{N-1} \frac{t^j}{j!} e^{A_1 t} A_1^j x_0, \quad 0 \leq t \leq t_1,
\]

\[
x(t) = e^{A_2 (t-t_1)} e^{A_1 t_1} x_0, \quad t_1 < t
\]

(10)

where the matrix exponential is defined as usual [30]:

\[
e^{A_1 t} = \sum_{j=0}^{\infty} \frac{t^j}{j!}
\]

\[
e^{A_2 t} = \sum_{j=0}^{\infty} \frac{t^j}{j!}
\]

(11)

The performance index (6) can be calculated without integration applying the algebraic Lyapunov matrix equation as follows.

Denote \( P_i \) (\( i = 1,2 \)) are the symmetric matrices solved from the Lyapunov matrix equations [14]:

\[
P_1 A_1 + A_1^T P_1 + Q = 0
\]

\[
P_2 A_2 + A_2^T P_2 + Q = 0
\]

(12)

Substituting \( Q \) from (12) into (6) we have:

\[
J = \int_0^t x^T \left( P_1 A_1 + A_1^T P_1 \right) x dt
\]

\[
-\int_0^t x^T \left( P_2 A_2 + A_2^T P_2 \right) x dt
\]

(13)

Noting the state space equations (8) we have:

\[
J = \int_0^t x^T P_1 A_1 x dt - \int_0^t x^T A_1^T P_1 x dt
\]

\[
-\int_0^t x^T P_2 A_2 x dt + \int_0^t x^T A_2^T P_2 x dt
\]

(14)

Calculating the integrals give:

\[
J = x_0^T P_1 x_0 + \int_0^t x^T e^{A_1 (t-t_1)} \left( P_2 - P_1 \right) e^{A_1 t_1} x dt
\]

\[
+ \lim_{t \to \infty} \int_0^t x^T e^{A_1 (t-t_1)} \left( -P_1 \right) e^{A_1 t_1} x dt
\]

(15)

Using the symbolic computation programs, such as MATLAB® Symbolic Math Toolbox, the matrix exponential (11) is determined by:

\[
e^{A_1 t} = e^{-\omega_n t} \begin{bmatrix}
cosh(\omega_n \sqrt{s_1^2 - 1} t) & \sinh(\omega_n \sqrt{s_1^2 - 1} t) \\
\sinh(\omega_n \sqrt{s_1^2 - 1} t) & \cosh(\omega_n \sqrt{s_1^2 - 1} t)
\end{bmatrix}
\]

\[
e^{A_2 t} = e^{-\omega_n t} \begin{bmatrix}
cosh(\omega_n \sqrt{s_2^2 - 1} t) & \sinh(\omega_n \sqrt{s_2^2 - 1} t) \\
\sinh(\omega_n \sqrt{s_2^2 - 1} t) & \cosh(\omega_n \sqrt{s_2^2 - 1} t)
\end{bmatrix}
\]

Note that:
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\[
\lim_{t \to \infty} e^{-\zeta_1 t} \cosh \left( \omega_0 \sqrt{\zeta_1^2 - 1} - t \right) \\
= \frac{1}{2} \lim_{t \to \infty} \left( e^{-\zeta_1 \sqrt{\zeta_1^2 - 1} t} + e^{-\zeta_1 \sqrt{\zeta_1^2 + 1} t} \right), \quad \zeta_1 > 1 \\
\lim_{t \to \infty} e^{-\zeta_1 t} \cos \left( \omega_0 \sqrt{1 - \zeta_1^2} t \right), \quad \zeta_2 \leq 1 \\
= 0
\]
\[
\lim_{t \to \infty} e^{-\zeta_2 t} \frac{\sinh \left( \omega_0 \sqrt{\zeta_2^2 - 1} t \right)}{\sqrt{\zeta_2^2 - 1}} \\
= \frac{1}{2} \lim_{t \to \infty} e^{-\zeta_2 t} \frac{\sinh \left( \omega_0 \sqrt{1 - \zeta_2^2} t \right)}{\sqrt{1 - \zeta_2^2}}, \quad \zeta_2 \leq 1 \\
= 0
\]
The limits (17) and (18) reduces (16) to:
\[
\lim_{t \to \infty} e^{\lambda t} = 0
\]
where 0 is the 2x2 zero matrix. The performance index (15) then is reduced to:
\[
J = x_0^T P_1 x_0 + x_0^T e^{A_{12} t} (P_2 - P_1) e^{A_{12} t} x_0
\]
Substituting (7) and (9) into the Lyapunov matrix equations (12) we have:
\[
\begin{bmatrix}
p_1 & p_2 \\
p_2 & p_3
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
-1 & -2 \zeta_1
\end{bmatrix}
+ \omega_0
\begin{bmatrix}
0 & -1 \\
1 & -2 \zeta_1
\end{bmatrix}
\begin{bmatrix}
p_1 & p_2 \\
p_2 & p_3
\end{bmatrix}
+ \frac{\omega_0}{2}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]
where
\[
P_1 =
\begin{bmatrix}
p_1 & p_2 \\
p_2 & p_3
\end{bmatrix}
\]
and \( p_1, p_2 \) and \( p_3 \) are three unknown variables. Expanding (20) give three equations:
\[
\begin{align*}
-2 p_2 + \frac{1}{2} &= 0 \\
p_1 - 2 \zeta_1 p_2 - p_3 &= 0 \\
2 p_2 - 4 \zeta_1 p_3 + \frac{1}{2} &= 0
\end{align*}
\]
which is solved:
\[
\begin{align*}
p_2 &= \frac{1}{4} \\
p_1 &= \frac{\zeta_1}{2} + \frac{1}{4 \zeta_1} \\
p_3 &= \frac{1}{4 \zeta_1}
\end{align*}
\]
or
\[
P_1 = \frac{1}{2}
\begin{bmatrix}
\zeta_1 + \frac{1}{2 \zeta_1} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2 \zeta_1}
\end{bmatrix}
\]
the same process gives:
\[
P_2 = \frac{1}{2}
\begin{bmatrix}
\zeta_2 + \frac{1}{2 \zeta_2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2 \zeta_2}
\end{bmatrix}
\]
Substituting \( P_1 \) and \( P_2 \) from (24), \( e^{A_{12} t} \) from (16) and \( x_0 \) from (5) into (19) we have:
\[
J = \frac{1}{2} \left( \zeta_1 + \frac{1}{2 \zeta_1} \right) \left( \frac{\cosh \left( \omega_0 \sqrt{\zeta_1^2 - 1} t \right)}{\sqrt{\zeta_1^2 - 1}} \right)^2
+ \frac{\zeta_2 - \zeta_1}{2} \left( \frac{\sinh \left( \omega_0 \sqrt{\zeta_1^2 - 1} t \right)}{\sqrt{\zeta_1^2 - 1}} \right) \left( \frac{1 - \frac{1}{2 \zeta_1 \zeta_2}}{2 \zeta_1 \zeta_2} \right)
\]
Some simplifications of (26) yields:
\[
J = \frac{1}{2} \left( \zeta_1 + \frac{1}{2 \zeta_1} \right) \left( \frac{\cosh \left( \omega_0 \sqrt{\zeta_1^2 - 1} t \right)}{\sqrt{\zeta_1^2 - 1}} \right)^2
+ \frac{\zeta_2 - \zeta_1}{2} \left( \frac{\sinh \left( \omega_0 \sqrt{\zeta_1^2 - 1} t \right)}{\sqrt{\zeta_1^2 - 1}} \right) \left( \frac{1 - \frac{1}{2 \zeta_1 \zeta_2}}{2 \zeta_1 \zeta_2} \right)
\]
In brief, the optimization problem is to find the switching time \( t \) and the switching damping ratio \( \zeta_1 \) and \( \zeta_2 \) to minimize \( J \) in (27)

III. OPTIMAL PASSIVE CONTROL

Denote \( \zeta_p \) as the nominal passive damping ratio, the passive controlled system gives:
\[
\zeta_1 = \zeta_2 = \zeta_p
\]
Using (28) in (27), the performance index reduces to:
\[
J_p = \frac{\zeta_p}{2} + \frac{1}{4 \zeta_p}
\]
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We found the minimum performance index is \( \frac{1}{\sqrt{2}} \approx 0.707 \) when the damping ratio is tuned to the optimal value of \( \frac{1}{\sqrt{2}} \).

IV. OPTIMAL SEMI-ACTIVE DAMPING CONTROL

The optimal condition \( \frac{\partial J}{\partial \zeta_1} = 0 \) applying to (27) can be simplified to:

\[
cosh\left(\omega_n \sqrt{\zeta_2 - \zeta_1^{opt}}\right)\left(\zeta_1 - \frac{1}{\zeta_2}\right) = 0
\]

where \( \zeta_1^{opt} \) is the optimal switching time. The equation (29) can be solved to obtain:

\[
\zeta_1^{opt} = \sqrt{\frac{2 - 2\zeta_1 - 2\zeta_2}{\zeta_2}} + 1
\]

(30)

The interesting result is that we can find optimal switching time in the case both the switching dampings \( \zeta_1 \) and \( \zeta_2 \) are fixed due to the practical conditions.

Use (29) in (27), after some manipulations, we reduce the performance index to:

\[
J_e = \frac{\zeta_1}{2} + \frac{1 - \zeta_2 - \zeta_1 e^{-2\zeta_1\omega_n t}}{4\zeta_1 \zeta_2}
\]

(31)

The plot of \( J_e \) versus \( \zeta_1 \) and \( \zeta_2 \) is shown in Fig. 2.

As seen in Fig. 2, the minimum of \( J_e \) is obtained when \( \zeta_1 \) tends to zero. It is intuitive because in the first time interval (\( 0 < t < t_1^{opt} \)), the damping should be as small as possible to allow the mass return to the equilibrium as quick as possible. In the second time interval (\( t_1^{opt} < t < \infty \)), the damping should be chosen optimally to prevent the overshoot. Therefore, first let \( \zeta_2 \) tend to zero. Using this limit in (30), we have the following optimal switching time (in the limit case):

\[
t_1^{opt} = \frac{\pi}{2\omega_n} \arccos \frac{1}{\sqrt{\zeta_2^2 + 1}}
\]

(32)

Using the limit \( (\zeta_1 \to 0) \) in (31) and noting (32) yield:

\[
\lim_{\zeta_1 \to 0} J_e = \frac{1}{2} \arccos \frac{1}{\sqrt{\zeta_2^2 + 1}} + \frac{1}{4\zeta_2}
\]

(33)

The vanishing of the derivative \( \partial \left( \lim_{\zeta_1 \to 0} J_e \right) / \partial \zeta_2 \) yields:

\[
\frac{\partial \left( \lim_{\zeta_1 \to 0} J_e \right)}{\partial \zeta_2} = 0 \Rightarrow \zeta_2^{opt} = 1
\]

(34)

Then the optimal switching time (32) reduces to:

\[
t_1^{opt} = \frac{\pi}{4\omega_n}
\]

(35)

Substituting (34) into (33) gives the final form of the minimized performance index of the one-time switching damping controller:

\[
J_e^{min} = \left( \lim_{\zeta_1 \to 0} J_e \right) = \frac{\pi}{8} + \frac{1}{4} = 0.6427
\]

(36)

It is interesting because although the performance index (27) is a complex function of three variables but the analytical minimum can be found. The simplest strategy to gain the minimum is as follows. First, the zero damping is used. At the time \( t_1^{opt} = \frac{\pi}{4\omega_n} \) (=1/8 of the undamped natural period), the damping ratio is turned to the critical value (=1) and held to the infinity.

V. CONCLUSIONS

This paper considers the switching damping problem of a free vibration bi-linear state space system. The normalized integrals of mechanical energy of passive and semi-active damping SDOF systems are minimized. The main result of this paper is to solve analytically the minimization problem of the semi-active damping SDOF. The values of the optimal performance index are 0.707, and 0.643 respectively for the passive and semi-active damping controlled systems. The optimal semi-active damping should switch the system from the undamped to the critically damped at the time of 1/8 of the undamped natural period.

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