Deflection and natural frequency of FGM plates with porosities by using Quasi-3D seven-unknowns higher-order shear deformation theory

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Abstract—This paper investigates the bending and vibration of an FGM rectangular plate with porosities under two porosity distribution patterns including even and uneven. Based on Hamilton’s principle, the governing equations of motion of novel Quasi-3D theory with seven-unknown variables are derived. By employing the Navier technique, the analytical solutions are obtained to determine the displacements, and natural frequency of simply supported rectangular plates. The comparison between the present results and the available ones in the literature is conducted to verify the accuracy of the proposed model. Moreover, numerical results are presented to examine the effects of material and geometric parameters on the deflections and natural frequency of FGM rectangular plates.

Keywords—Bending, natural frequency, navier technique, fgm with porosity, quasi-3d theory.

I. INTRODUCTION

Functionally graded materials (FGMs) are a widely used category of composites in different engineering aspects such as aerospace, fusion reactors, and the nuclear industry. They possess a unique feature wherein the material characteristics change gradually across the thickness, effectively mitigating the stress concentration issues commonly encountered in traditional laminated composites. Moreover, in the case of a typical FGM composed of a blend of metals and ceramics, it accomplishes a dual objective by simultaneously attaining the desired attributes of both materials: metals provide high strength and fracture toughness, while ceramics offer corrosion resistance and temperature resilience. The studies related to the static and dynamic behavior of FGM plates or beams, have attracted scientists in recent years and can be found in the existing resources [1-7]. During the manufacturing of FGMs, micro voids or porosity can occur within the material. This is primarily due to the fact that the constituent materials of FGMs have varying solidification temperatures, which result in the development of these micro voids or porosities during the fabrication process. The existence of these micro voids or porosity within the material diminishes the plate's mechanical strength, potentially resulting in structural failure [8].

Vibration problems of perfect and imperfect FGM plates (FGM with porosities, porous FGM - named FGP) have received considerable attention over the past decades. Various effective plate theories and numerical methods focusing on this topic have been developed. The classical plate theory has been applied to perform vibration analysis of the plate [9-11]. The assumptions of CPT are quite satisfactory for calculating the critical buckling load for thin plates. However, the accuracy of results based on CPT may lead to inaccuracy for moderate thick to thick plate. This inaccuracy is observed due to the negligence of shear deformation in the transverse direction while this effect plays an important role in thick plate. Numerous plate theories have been developed by various researchers based on the shear deformation effect. Among them, the transverse shear deformation effect is included in FSDT (first-order shear deformation theory) which is known as the Reissner-Mindlin theory [12-14]. In term of FSDT it is necessary to use the shear correction factors, but determining them is difficult because it depends on many parameters such as materials, geometry, boundary condition, etc. So, HSDTs (higher-order shear deformation theories) have been
where: $\epsilon_0$ is the porosity coefficient ($0 \leq \epsilon_0 << 1$). It is observed that when $\epsilon_0 = 0$ the considered material is perfect FGM. $P_c$ and $P_m$ depict material properties of ceramic and metal, respectively. $\rho$ is the power-law index ($\rho \geq 0$).

### III. DISPLACEMENT, STRAIN AND STRESS FIELDS

The refined quasi-3D HSDT used in this study originated from eleven unknown HSDT and also satisfies free transverse shear stresses in the top and bottom surfaces of the plate. The displacement field is given as [24]:

$$u = u_0(x, y, t) + z\theta_x(x, y, t) - \frac{z^2}{2}\frac{\partial^2\theta_x}{\partial x^2}$$
$$v = v_0(x, y, t) + z\theta_y(x, y, t) - \frac{z^2}{2}\frac{\partial^2\theta_y}{\partial y^2},$$
$$w = w_0(x, y, t) + z\theta_z(x, y, t) - z^2w_0(x, y, t)$$

where: $c_0 = \frac{4}{h^2}$; $u_0, v_0, w_0$ are displacements of any point in the mid-plane with along $(x, y, z)$ axes; $\theta_x, \theta_y$ are rotations of normal $y, x$ axes respectively; $\theta_z, w_0$ are high-order terms in Taylor series expansion of displacement functions.

The linear strain components attained from the strain-displacement relations are given as follows:

$$\epsilon_x = \epsilon^0_x + z\kappa^0_x + z^2\epsilon^*_x + z^3\kappa^*_x; \quad i = x, y$$
$$\epsilon_z = \epsilon^0_z + z\kappa^0_z;$$
$$\gamma_{xy} = \gamma^0_{xy} + z\gamma^0_{xy} + z^2\gamma^*_{xy} + z^3\kappa^*_y;$$
$$\gamma_{xz} = \gamma^0_{xz} + z\gamma^0_{xz} + z^2\gamma^*_{xz} + z^3\kappa^*_x;$$

in which: $\epsilon^0_x = \frac{\partial u_0}{\partial x}; \kappa^0_x = \frac{\partial \theta_x}{\partial x}; \epsilon^*_x = \frac{1}{2}
\frac{\partial^2\theta_x}{\partial x^2};$
$$\kappa^*_x = \left[-\frac{1}{3}c_0\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \theta_x}{\partial x}\right) + \frac{\partial w_0^*}{\partial x}\right];$$
$$\epsilon^0_y = \frac{\partial v_0}{\partial y}; \kappa^0_y = \frac{\partial \theta_y}{\partial y}; \epsilon^*_y = -\frac{1}{2}\frac{\partial^2\theta_y}{\partial y^2};$$
$$\kappa^*_y = \left[-\frac{1}{3}c_0\left(\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial \theta_y}{\partial y}\right) + \frac{\partial w_0^*}{\partial y}\right];$$
$$\epsilon^0_z = \theta_z; \kappa^0_z = 2w_0^*; \gamma^0_{xy} = -\frac{\partial \theta_x}{\partial y};$$
$$\gamma^0_{xz} = -\frac{\partial \theta_x}{\partial x} + \frac{\partial \theta_y}{\partial x};$$

Taking into account the power law for volume fraction, the modified effective mechanical properties of an FGP can be written as [22, 23]:

\text{FGP-I: } P(z) = P_m + (P_c - P_m) \left(\frac{z}{h} + \frac{1}{2}\right)^\rho

\text{FGP-II: } P(z) = P_m + (P_c - P_m) \left(\frac{z}{h} + \frac{1}{2}\right)^\rho

\text{(1)}

\text{(2)}
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\[ \kappa_{xy} = -\frac{1}{3} \left[ c_0 \left( \frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial^2 \theta_x}{\partial y} + \frac{\partial^2 \theta_y}{\partial x} \right) + 2 \frac{\partial^2 w_0}{\partial x \partial y} \right]; \]

\[ \gamma_{x}^0 = \frac{\partial \theta_x}{\partial x} + \frac{\partial w_0}{\partial y}; \gamma_{y}^0 = \frac{\partial \theta_y}{\partial y} + \frac{\partial w_0}{\partial x}. \]

Kinematic relations are given according to Hooke's law:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\sigma_{xy} \\
\sigma_{xz} \\
\sigma_{yz}
\end{bmatrix} = \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\
Q_{21} & Q_{22} & Q_{23} & 0 & 0 & 0 \\
Q_{31} & Q_{32} & Q_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & Q_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & Q_{66} & 0 \\
0 & 0 & 0 & 0 & 0 & Q_{55}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix},
\]

where:

\[ Q_{11} = Q_{22} = Q_{33} = \frac{(1 - \nu)E}{(1 + \nu)(1 - 2\nu)}; \]

\[ Q_{12} = Q_{13} = Q_{23} = Q_{21} = Q_{31} = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}; \]

\[ Q_{44} = Q_{55} = Q_{66} = \frac{E}{2(1 + \nu)}. \]

IV. EQUATIONS OF MOTION

The equations of motion are derived by utilizing Hamilton’s principle and are expressed as follows [24]:

\[ N_{x,x} + N_{xy,y} = I_0 \ddot{\theta}_x - \frac{L_s}{3} \left( c_0 \dot{w}_{0,x} + w^*_0 \right); \]

\[ N_{xy,x} + N_{y,y} = I_0 \ddot{\theta}_y - \frac{L_s}{3} \left( c_0 \dot{w}_{0,y} + w^*_0 \right); \]

\[ \frac{c_0}{3} \left( M_{x,x} + 2M_{xy,y} + 3M_{x,y} \right) + R_{x,x} + R_{x,y} + q_z^+ = I_0 \ddot{w}_0 + I_2 \ddot{\theta}_x + I_2 \ddot{\theta}_y + \frac{L_s c_0}{3} \left( u_{0,y} + \dot{v}_{0,y} \right) - I_6 \frac{c_0}{6} \nabla^2 \ddot{z}; \]

\[ + \frac{c_0 I_4}{3} \left( \theta_{x,x} + \theta_{x,y} \right) - \frac{I_6 c_0^2}{9} \nabla^2 w_0^* - \frac{I_6 c_0}{9} \nabla^2 w_{0,y}^*; \]

\[ P_{x,x} + P_{x,y} - R_j = J_4 \ddot{\theta}_x + K_4 \ddot{\theta}_y - \frac{J_4}{3} \dot{\theta}_{x,y} - \frac{c_0 J_4}{3} \dot{w}_{0,x} - \frac{J_4}{3} \dot{w}_{0,y}^*; \]

\[ P_{y,x} + P_{y,y} - R_j = J_4 \ddot{\theta}_x + K_4 \ddot{\theta}_y - \frac{J_4}{3} \dot{\theta}_{x,y} - \frac{c_0 J_4}{3} \dot{w}_{0,x} - \frac{J_4}{3} \dot{w}_{0,y}^*; \]

\[ \frac{1}{2} \left( N_{x,x} + 2N_{xy,y} + N_{x,y} \right) - N_z + q_z^+ \frac{h}{2} = I_0 \ddot{w}_0 + I_2 \ddot{\theta}_x + I_2 \ddot{\theta}_y + \frac{L_s c_0}{3} \left( u_{0,x} + \dot{v}_{0,x} \right) + I_4 \ddot{w}_0^* \]

\[ + \frac{J_3}{2} \left( \ddot{\theta}_{x,x} + \ddot{\theta}_{x,y} \right) - \frac{I_4}{4} \nabla^2 \ddot{z} + \frac{I_6 c_0}{6} \nabla^2 w_0^* - \frac{I_6}{6} \nabla^2 w_{0,y}^*; \]

\[ \frac{1}{2} \left( M_{x,x} + 2M_{xy,y} + 3M_{x,y} \right) - 2M_z + q_z^+ \frac{h^2}{4} = I_2 \ddot{w}_0 + I_2 \ddot{\theta}_x + I_2 \ddot{\theta}_y + \frac{L_s c_0}{3} \left( u_{0,x} + \dot{v}_{0,x} \right) + I_4 \ddot{w}_0^* \]

\[ + \frac{J_3}{3} \left( \ddot{\theta}_{x,x} + \ddot{\theta}_{x,y} \right) - \frac{I_4}{6} \nabla^2 \ddot{z} - \frac{I_6 c_0}{9} \nabla^2 w_0^* - \frac{I_6}{9} \nabla^2 w_{0,y}^* \]

where: \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is Laplacian operator in two-dimensional Cartesian coordinate system.

The stress resultants are defined by:

\[ \begin{bmatrix}
N_x \\
N_y \\
N_z \\
N_{xy} \\
N_{xz} \\
N_{yz}
\end{bmatrix} = \begin{bmatrix}
\frac{h^2}{2} \\
\frac{h^2}{2} \\
\frac{h}{2} \\
\frac{h^2}{2} \\
\frac{h^2}{2} \\
\frac{h}{2}
\end{bmatrix} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\sigma_{xy} \\
\sigma_{xz} \\
\sigma_{yz}
\end{bmatrix} \begin{bmatrix}
1 \\
z \\
z^2 \\
z^3
\end{bmatrix} dz; \]

\[ \begin{bmatrix}
N_{x,x} \\
N_{y,y} \\
N_{x,y}
\end{bmatrix} = \begin{bmatrix}
\frac{h^2}{2} \\
\frac{h^2}{2} \\
\frac{h^2}{2} \\
\frac{h^2}{2} \\
\frac{h^2}{2} \\
\frac{h^2}{2}
\end{bmatrix} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\sigma_{xy} \\
\sigma_{xz} \\
\sigma_{yz}
\end{bmatrix} \begin{bmatrix}
1 \\
z^2 \\
z^3
\end{bmatrix} dz; \]

\[ \begin{bmatrix}
R_x \\
R_y \\
R_z
\end{bmatrix} = \begin{bmatrix}
\frac{h^2}{2} \\
\frac{h^2}{2} \\
\frac{h^2}{2}
\end{bmatrix} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z
\end{bmatrix} \begin{bmatrix}
1 \\
z^2 \\
z^3
\end{bmatrix} dz; \]

The mass inertias are defined by:

\[ I_i = \int \rho(z) dz; i = 0 \div 6; \]

\[ J_i = \int \rho(z) dz; i = 1,3,4; \]

\[ K_2 = I_2 - \frac{c_0}{3} I_4 + \frac{c_0^2}{9} I_6 \]

V. NAVIER’S SOLUTION

Consider a simply supported rectangular FGM plates with length \( a \), width \( b \) under the transverse distributed load. The associated simply supported boundary conditions are as follows:

At edge \( x = 0 \) and \( x = a \):

\[ v_0 = w_0 = \theta_y = \theta_z = w^*_0 = M_x = M^*_x = 0 \]
At edge $y = 0$ and $y = b$:

$$u_0 = w_0 = \theta_x = \theta_z = w_0^* = M_y = M_y^* = 0$$

Following Navier’s solution procedure, the displacement variables are chosen to satisfy the above boundary condition in the following forms:

$$u_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{0mn}(t) \cos \alpha x \sin \beta y;$$
$$v_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{0mn}(t) \sin \alpha x \cos \beta y;$$
$$w_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{0mn}(t) \sin \alpha x \sin \beta y;$$
$$\theta_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{xmn}(t) \cos \alpha x \sin \beta y;$$
$$\theta_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{ymn}(t) \sin \alpha x \cos \beta y;$$
$$\theta_z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{zmn}(t) \sin \alpha x \sin \beta y;$$
$$w_0^* = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{0mn}^* \sin \alpha x \cos \beta y$$

where: $u_{0mn}, v_{0mn}, w_{0mn}, \theta_{xmn}, \theta_{ymn}, \theta_{zmn}, w_{0mn}^*$ are the unknown coefficients; $\alpha = \frac{m\pi}{a}, \beta = \frac{n\pi}{b}$.

The applied transverse load $q_z^+(x, y, t)$ is also expanded in double-Fourier sine series:

$$q_z^+ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn}(t) \sin \alpha x \sin \beta y$$

The coefficients $q_{mn}$ are given below for any typical loads:

$$q_{mn}(t) = \frac{4}{ab} \int_0^a \int_0^b q_z^+(x, y, t) \sin \alpha x \sin \beta y \, dx \, dy$$

for uniformly distributed load, $q_z^+ = q_0(t)$:

$$q_{mn}(t) = \frac{16q_0(t)}{\pi^2 mn}; \quad m, n = 1, 3, 5, \cdots$$

for sinusoidal distributed load, $q_z^+ = q_0(t) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$:

$$q_{mn}(t) = q_0(t); \quad m = n = 1$$

Substituting Eqs. (10) and (11) into Eq. (6), the closed-form solutions can be obtained from:

$$KQ + M\ddot{Q} = F$$

(15)

where: $Q = \left[ u_{0mn}, v_{0mn}, w_{0mn}, \theta_{xmn}, \theta_{ymn}, \theta_{zmn}, w_{0mn}^* \right]^T$,

$$F = \left\{ 0 \ 0 \ q_{mn} \ 0 \ 0 \ \frac{h}{2} q_{mn} \ \frac{h^2}{4} q_{mn} \right\}^T$$

the elements of matrix $K, M$ are defined in the Appendix.

The system of Eq. (15) may be used to obtain the solutions of the bending problems of the FGP plates by dropping all the inertia terms:

$$KQ = F$$

(16)

The solutions of the free vibration problems of the plates by removing transverse load. The displacement vector is assumed in the form $Q = Q_0 e^{i \omega t}$, the natural frequency of the FGP plate is determined by solving the eigenvalue problem below:

$$(K - \omega^2 M)Q = 0$$

(17)

VI. NUMERICAL RESULTS AND DISCUSSION

With the obtained analytical solutions, the homemade Matlab’s codes are employed to conduct the numerical examples. The simply supported rectangular FGP plates with the ceramic enriched top ($\text{Al}_2\text{O}_3$) and the metal enriched bottom (Al). The input data are: $E_c = 380$ GPa, $v_c = 0.3$, $\rho_c = 3800$ kg/m$^3$, $E_m = 70$ GPa, $v_m = 0.3$, $\rho_m = 2702$ kg/m$^3$. The Poisson ratio $\nu$ is assumed to be constant ($\nu = 0.3$). For convenience, the following non-dimensional forms [25, 26] are used:

$$\bar{w} = \frac{10}{\sqrt{E_m \rho_m}} \bar{q}_0 \left( \frac{a}{2}, \frac{b}{2}, 0 \right) \bar{E}_h^3 \bar{q}_0 \bar{a}^3; \quad \bar{\omega} = \omega h \sqrt{\frac{\rho_m}{E_m}}$$

(18)

A. Validation study

To indicate the accuracy and efficiency of the proposed quasi-3D HSDT, the numerical results are compared with those calculated by using the 3D, several existing quasi-3D and HSDT theories.

For validation of deflection, let’s consider a square FGM plate ($\text{Al}/\text{Al}_2\text{O}_3$) under a sinusoidally distributed load $q = q_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$. Table I presents the non-dimensional central deflection $\bar{w}$ for various side-to-thickness ratios $a/h$. Present results are compared with those of Nguyen and Nguyen-Xuan [27] using 3D theory, of Neves et al. [28], Akavci and Tanrikulu [29], and Farzam-Rad et al. [25] using quasi-3D theories with volume fraction index $p = 1$. Excellent agreement can be observed between the results.
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Table I. Validation of non-dimensional central deflection of perfect FGM plates

<table>
<thead>
<tr>
<th>Model</th>
<th>( \varepsilon_e )</th>
<th>( \omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( a/h = 4 )</td>
<td>( a/h = 10 )</td>
</tr>
<tr>
<td>3D [27]</td>
<td>( \neq 0 )</td>
<td>0.7171</td>
</tr>
<tr>
<td>Quasi-3D [28]</td>
<td>( \neq 0 )</td>
<td>0.7020</td>
</tr>
<tr>
<td>Quasi-3D [29]</td>
<td>( \neq 0 )</td>
<td>0.6908</td>
</tr>
<tr>
<td>Quasi-3D [25]</td>
<td>( \neq 0 )</td>
<td>0.6911</td>
</tr>
<tr>
<td>Quasi-3D [28]</td>
<td>( \neq 0 )</td>
<td>0.7308</td>
</tr>
<tr>
<td>Quasi-3D [29]</td>
<td>( \neq 0 )</td>
<td>0.7282</td>
</tr>
<tr>
<td>Quasi-3D [25]</td>
<td>( \neq 0 )</td>
<td>0.7284</td>
</tr>
<tr>
<td>Present</td>
<td>( \neq 0 )</td>
<td><strong>0.7171</strong></td>
</tr>
</tbody>
</table>

Discrepancy (%): \( \neq 0 \) | 0.00 | 0.01 | 0.01 |

*Discrepancy between present and 3D results [27].

For validation of the natural frequencies of FGM plate, the square perfect FGM plate (Al/Al\(_2\)O\(_3\)) is considered. The non-dimensional fundamental natural frequency \( \tilde{\omega} \) of FGM plate with different volume fraction index \( p \) and side-to-thickness ratio \( h/a \) are presented in Table II. The present results are compared with those given by Jin [30] using 3D theory, by Mantari et al. [31], Farzam-Rad et al. [25], and Shahsavari et al. [20] using quasi-3D theories. The obtained results show high agreement with the results of previous studies.

Table II. Validation of non-dimensional fundamental natural frequency of perfect FGM square plates

<table>
<thead>
<tr>
<th>( h/a )</th>
<th>Model</th>
<th>( p = 1 )</th>
<th>( p = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3D exact [30]</td>
<td>0.0870</td>
<td>0.0789</td>
</tr>
<tr>
<td></td>
<td>Quasi-3D [31]</td>
<td>0.0882</td>
<td>0.0806</td>
</tr>
<tr>
<td></td>
<td>Quasi-3D [25]</td>
<td>0.0882</td>
<td>0.0806</td>
</tr>
<tr>
<td></td>
<td>Quasi-3D [20]</td>
<td>0.0882</td>
<td>0.0806</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td><strong>0.0870</strong></td>
<td><strong>0.0789</strong></td>
</tr>
<tr>
<td>Discrepancy (%)</td>
<td>0.05</td>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>3D exact [30]</td>
<td>1.4687</td>
<td>1.3095</td>
</tr>
<tr>
<td></td>
<td>Quasi-3D [31]</td>
<td>1.4774</td>
<td>1.3219</td>
</tr>
<tr>
<td></td>
<td>Quasi-3D [25]</td>
<td>1.4788</td>
<td>1.3226</td>
</tr>
<tr>
<td></td>
<td>Quasi-3D [20]</td>
<td>1.4772</td>
<td>1.3218</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td><strong>1.4716</strong></td>
<td><strong>1.3136</strong></td>
</tr>
<tr>
<td>Discrepancy (%)</td>
<td>0.20</td>
<td>0.31</td>
<td></td>
</tr>
</tbody>
</table>

*Discrepancy between present and 3D results [30].

Table III presents the non-dimensional central deflection of the square FGP plates with various side-to-thickness ratios \( a/h \) and volume fraction index under the uniformly distributed load \( q = q_0 \). Two porosity patterns and two values of porosity coefficients are considered. These results again show small discrepancies compared to the results of Dhuria et al. [32] using HSDT-5 (5 displacement unknowns).

Table III. Validation of non-dimensional central deflection of square FGP plates

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \varepsilon_e )</th>
<th>( a/h = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even</td>
<td>Dhuria et al. [32]</td>
<td>Present</td>
</tr>
<tr>
<td>Uneven</td>
<td>Dhuria et al. [32]</td>
<td>Present</td>
</tr>
</tbody>
</table>

\[ 0.1 \quad 0 \quad 0.5881 \quad 0.5799 \quad 0.5881 \quad 0.5799 \]
\[ 0.2 \quad 0.6769 \quad 0.6677 \quad 0.6141 \quad 0.6053 \]
\[ 0.5 \quad 0 \quad 0.8053 \quad 0.7983 \quad 0.8053 \quad 0.7983 \]
\[ 0.2 \quad 0.9968 \quad 0.9889 \quad 0.8578 \quad 0.8504 \]
\[ 1 \quad 0 \quad 1.0410 \quad 1.0357 \quad 1.0410 \quad 1.0357 \]
\[ 0.2 \quad 1.4328 \quad 1.4267 \quad 1.1424 \quad 1.1371 \]

\[ 0.1 \quad 0 \quad 0.5162 \quad 0.5142 \quad 0.5162 \quad 0.5142 \]
\[ 0.2 \quad 0.5946 \quad 0.5924 \quad 0.5354 \quad 0.5333 \]
\[ 0.5 \quad 0 \quad 0.7147 \quad 0.7130 \quad 0.7147 \quad 0.7130 \]
\[ 0.2 \quad 0.8887 \quad 0.8868 \quad 0.7561 \quad 0.7544 \]
\[ 1 \quad 0 \quad 0.9279 \quad 0.9266 \quad 0.9279 \quad 0.9266 \]
\[ 0.2 \quad 1.2913 \quad 1.2900 \quad 1.0115 \quad 1.0103 \]

Table IV presents non-dimensional natural frequencies for four vibrational modes of FGP square plates \((a/h = 20, p = 1)\). Uneven porosity distribution (FGP-II), two values of porosity coefficients \( \varepsilon_0 = 0.1 \) and \( \varepsilon_0 = 0.2 \) are considered. From this table, it is observed that the obtained results are almost identical in comparison with those given by Merdaci et al. [18] using HSDT-4 (four displacement unknowns).

Table IV. Validation of non-dimensional natural frequencies of FGP square plates FGP

<table>
<thead>
<tr>
<th>( \varepsilon_0 )</th>
<th>Source</th>
<th>Mode ((m, n))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( (1,1) )</td>
<td>( (1,2) )</td>
</tr>
<tr>
<td>0.1</td>
<td>Merdaci et al. [18]</td>
<td>0.0224</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td><strong>0.0223</strong></td>
</tr>
<tr>
<td>0.2</td>
<td>Merdaci et al. [18]</td>
<td>0.0225</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td><strong>0.0225</strong></td>
</tr>
</tbody>
</table>

B. Parametric study

The effect of material parameters including volume fraction index \( p \), porosity coefficient \( \varepsilon_0 \), and porosity distribution patterns on nondimensional deflection and fundamental natural frequency of FGP plate \((a = b = 10h)\) is depicted in Figs 2 and 3.
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It can be seen that, for each porosity coefficient, when increasing volume fraction index \( p \) (means increasing the metal content in the FGP), the stiffness of the plate structure decreases, leading to an increase in the central deflection, and decrease in the fundamental frequency. Moreover, increasing the porosity coefficient will increase the central deflection, and as \( p \) becomes larger, the influence of porosity coefficient \( e_0 \) becomes more pronounced. Perfect FGP (without porosities) plates always have the smallest central deflection; imperfect FGP plates with uneven porosity distribution have smaller central deflection compared to even porosity distribution. The influence of the porosity coefficient and the porosity distribution patterns on the natural frequency is quite complex and depends on the value of the volume fraction index \( p \). It is due to the correlation between the stiffness of the plate and the mass inertia effect.

VII. CONCLUSION

In this study, the Quasi-3D seven unknowns HSCT, which satisfies stress-free on the top and bottom surfaces is employed to predict the deflection and natural frequency of FGP plates. Power law FGM with two porosity distribution patterns are considered. The analytical solutions using Navier solution for a simply supported rectangular plate and the homemade Matlab’s codes have been built and verified with the results in existing literature. The numerical examples are conducted to evaluate the effect of material and geometry parameters on deflection and natural frequency of the FGP plate. In general, the existence of porosities reduces deflection as well as natural frequency. The even porosity pattern makes the plate softer than uneven porosity pattern.

ACKNOWLEDGMENT

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 107.02-2021.16.

REFERENCES

Deflection and natural frequency of FGM plates with porosities by using Quasi-3D seven-unknowns …


APPENDIX. THE STIFFNESS MATRIX $K$, MASS MATRIX $M$

$$K = \begin{bmatrix}
s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} & s_{17} \\
s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} & s_{27} \\
s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} & s_{37} \\
s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} & s_{47} \\
s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} & s_{57} \\
s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} & s_{67} \\
s_{71} & s_{72} & s_{73} & s_{74} & s_{75} & s_{76} & s_{77}
\end{bmatrix}$$

$$M = \begin{bmatrix}
m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} & m_{17} \\
m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} & m_{27} \\
m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} & m_{37} \\
m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} & m_{47} \\
m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & m_{56} & m_{57} \\
m_{61} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66} & m_{67} \\
m_{71} & m_{72} & m_{73} & m_{74} & m_{75} & m_{76} & m_{77}
\end{bmatrix}$$

Coefficients of matrix $K$:

$s_{11} = A_1 \alpha^2 + A_2 \beta^2$;  $s_{12} = s_{21} = (A_2 + A_4) \alpha \beta$

$s_{13} = s_{31} = \left(\frac{D_{11} c_0}{3} - \frac{D_{12} c_0}{3} \right) \alpha \beta^2$;

$s_{14} = s_{41} = \left(\frac{B_{11} - \frac{D_{11} c_0}{3}}{3} \right) \alpha^2 + \left(\frac{B_{44} - \frac{D_{44} c_0}{3}}{3} \right) \beta^2$;

$s_{15} = s_{51} = \left(\frac{B_{12} + B_{44} - \frac{D_{12} c_0}{3} - \frac{D_{44} c_0}{3}}{3} \right) \alpha \beta$;

$s_{16} = s_{61} = -\frac{C_{11}}{2} \alpha^3 - A_3 \alpha^2 \left(-\frac{C_{12} + C_{44}}{2} \right) \alpha \beta^2$;

$s_{17} = s_{71} = \frac{-D_{11}}{3} \alpha^3 - 2B_{11} \alpha - \left(\frac{D_{12} + 2D_{44}}{3} \right) \alpha \beta^2$;

$s_{22} = A_{44} \alpha^2 + A_{22} \beta^2$;

$s_{23} = s_{32} = \left(-\frac{D_{21} + 2D_{44}}{3} \right) c_0 \alpha^2 \beta - \frac{D_{22} c_0}{3} \beta^2$;

$s_{24} = s_{42} = \left(\frac{B_{21} + B_{44} - \frac{D_{21} c_0}{3} - \frac{D_{44} c_0}{3}}{3} \right) \alpha \beta$;

$s_{25} = s_{52} = \left(B_{44} - \frac{D_{44} c_0}{3} \right) \alpha^2 + \left(\frac{B_{22} - \frac{D_{22} c_0}{3}}{3} \right) \beta^2$;

$s_{26} = s_{62} = \left(-\frac{C_{21} + C_{44}}{2} \right) \alpha^2 \beta - A_{55} \beta - \frac{C_{22}}{2} \beta^3$;

$s_{27} = s_{72} = \left(-\frac{D_{21} + 2D_{44}}{3} \right) \alpha^2 \beta - 2B_{23} \beta - \frac{D_{22}}{3} \beta^3$;

$s_{33} = \frac{G_{11} c_0^2}{9} \alpha^4 + \left(A_{55} + E_{55} c_0^2 - 2C_{55} c_0 \right) \alpha^2$;

$s_{34} = \frac{G_{12} + G_{21} + 4G_{44}}{9} c_0^2 \alpha^2 \beta^2$;

$s_{35} = \frac{G_{22} c_0^2}{9} \beta^4$;

$s_{36} = \frac{G_{12} c_0^2}{3} - \frac{G_{13} c_0^2}{3} - \frac{2G_{44} c_0^2}{9} \alpha \beta^2$;

$s_{37} = \frac{B_{12}^2 c_0}{6} \alpha^4 + \frac{B_{44} c_0^2}{3} \alpha^2$;

$s_{36} = \frac{F_{12} c_0 c_0}{6} \alpha^4 + \frac{F_{21} c_0^2}{3} \alpha^2$;

$s_{37} = \frac{F_{11} c_0}{6} \alpha^4 + \frac{F_{22} c_0^2}{6} \beta^4$;

$s_{38} = \frac{G_{11} c_0^2}{9} \alpha^4 + \frac{2E_{55} c_0^2}{3} \alpha^2$;

$s_{39} = \frac{G_{12} + G_{21} + 4G_{44}}{9} c_0^2 \alpha^2 \beta^2$;

$s_{40} = \frac{2E_{23} c_0^2}{3} \beta^2 + \frac{G_{22} c_0^2}{9} \beta^4$;

$s_{41} = \frac{G_{22} c_0^2}{9} \beta^4$;

$s_{42} = \frac{G_{12} + G_{21} + 4G_{44}}{9} c_0^2 \alpha^2 \beta^2$;

$s_{43} = \frac{2E_{23} c_0^2}{3} \beta^2 + \frac{G_{22} c_0^2}{9} \beta^4$;

$s_{44} = \frac{G_{22} c_0^2}{9} \beta^4$;
\[ s_{44} = \left( \frac{G_1C_0}{9} - \frac{2E_1c_0}{3} + C_{11} \right) \alpha^2 \]
\[ + \left( \frac{G_{44}c_0^2}{9} - \frac{2E_{44}c_0}{3} + C_{44} \right) \beta^2 \]
\[ + E_{55}c_0^2 - 2C_{55}c_0 + A_{55}; \]
\[ s_{45} = s_{44} = \left( C_{12} + C_{44} + \frac{G_2c_0}{9} + \frac{G_{44}c_0}{9} \right) \alpha \beta; \]
\[ s_{46} = s_{44} = -\left( \frac{D_{12}}{2} - \frac{F_{12}c_0}{6} \right) \alpha^3 + \left( \frac{D_{12}c_0}{3} - B_{11} \right) \alpha \]
\[ - \left( \frac{D_{21}}{2} + \frac{D_{44}}{2} - \frac{F_{21}c_0}{6} - \frac{F_{44}c_0}{3} \right) \alpha \beta^2; \]
\[ s_{47} = s_{44} = \left( \frac{E_{12} - 2E_{44}}{3} - \frac{G_{23}c_0}{9} - \frac{2G_{44}c_0}{9} \right) \alpha \beta^2; \]
\[ s_{55} = \left( \frac{G_{44}c_0^2}{9} - \frac{2E_{44}c_0}{3} + C_{44} \right) \alpha^2 \]
\[ + \left( \frac{G_{23}c_0^2}{9} - \frac{2E_{23}c_0}{3} + C_{22} \right) \beta^2 \]
\[ + E_{56}c_0^2 - 2C_{56}c_0 + A_{56}; \]
\[ s_{56} = s_{55} = \left( \frac{D_{12}}{2} + \frac{D_{44}}{2} - \frac{F_{12}c_0}{6} - \frac{F_{44}c_0}{3} \right) \alpha^2 \beta \]
\[ + \left( \frac{D_{23}c_0}{3} - B_{32} \right) \beta - \left( \frac{D_{21}}{2} - \frac{F_{21}c_0}{6} \right) \beta^3; \]
\[ s_{57} = s_{55} = \left( \frac{E_{12} - 2E_{44}}{3} - \frac{G_{23}c_0}{9} - \frac{2G_{44}c_0}{9} \right) \alpha \beta^2 \]
\[ + \left( \frac{2E_{23}c_0}{3} - 2C_{23} \right) \beta - \left( \frac{E_{21}}{3} - \frac{G_{22}c_0}{9} \right) \beta^3; \]
\[ s_{66} = \frac{E_{41}}{4} \alpha^4 + \left( \frac{C_{13} + C_{31}}{2} \right) \alpha^2 \left( \frac{E_{12} + E_{21} + E_{44}}{4} \right) \alpha^2 \beta^2 \]
\[ + \left( C_{33} + C_{32} \right) \beta^2 + \left( \frac{E_{22}}{4} \right) \beta^3 + A_{33}; \]
\[ s_{67} = s_{66} = \frac{F_{13}}{6} \alpha^4 + \left( \frac{D_{13} + D_{31}}{2} \right) \alpha^2 \]
\[ + \left( \frac{F_{12}}{6} + \frac{F_{21} + 2F_{44}}{3} \right) \alpha^2 \beta^2 \]
\[ + \left( D_{23} + D_{32} \right) \beta^2 + \frac{F_{22}}{6} \beta^4 + 2B_{33}; \]
\[ s_{77} = \frac{G_{11}}{9} \alpha^4 + \left( \frac{2E_{43} + 2E_{31}}{3} \right) \alpha^2 \]
\[ + \left( \frac{G_{12} + G_{21} + 4G_{44}}{9} \right) \alpha^2 \beta^2 \]
\[ + \left( \frac{2E_{23} + 2E_{32}}{3} \right) \beta^2 + \frac{G_{22}}{9} \beta^4 + 4C_{33}; \]
\[ A_j = \int_{-h/2}^{h/2} Q_j dz; \]
\[ \left( B_j, C_j, D_j \right) = \int_{-h/2}^{h/2} Q_j \left( z, z^2, z^3 \right) dz; \]
\[ \left( E_j, F_j, G_j \right) = \int_{-h/2}^{h/2} Q_j \left( z^4, z^5, z^6 \right) dz; \]
\[ (i, j, k, l) = 1, 2, 3, 4, 5, 6 \]

Coefficients of matrix \( M \):
\[ m_{11} = m_{22} = I_0; \]
\[ m_{13} = m_{31} = -\frac{ac_0}{3} I_3; \]
\[ m_{44} = m_{44} = I_3; \]
\[ m_{23} = m_{32} = -\frac{\beta c_0}{3} I_3; \]
\[ m_{56} = m_{56} = \frac{\alpha}{2} I_2; \]
\[ m_{25} = m_{52} = I_1; \]
\[ m_{26} = m_{62} = -\frac{\beta}{2} I_2; \]
\[ m_{27} = m_{72} = -\frac{\beta}{3} I_3; \]
\[ m_{33} = I_0 + \frac{\left( a^2 + \beta^2 \right) c_0}{9} I_6; \]
\[ m_{34} = m_{43} = \frac{-ac_0}{3} I_4; \]
\[ m_{35} = m_{53} = \frac{-\beta c_0}{3} I_4; \]
\[ m_{36} = m_{63} = I_1 + \frac{\left( a^2 + \beta^2 \right) c_0}{9} I_5; \]
\[ m_{37} = m_{73} = I_2 + \frac{\left( a^2 + \beta^2 \right) c_0}{9} I_6; \]
\[ m_{44} = m_{55} = K_2; \]
\[ m_{46} = m_{64} = -\frac{\alpha}{2} I_3; \]
\[ m_{47} = m_{74} = -\frac{\beta}{2} I_3; \]
\[ m_{48} = m_{84} = \frac{-ac_0}{2} I_3; \]
\[ m_{56} = m_{65} = \frac{-\beta c_0}{2} I_3; \]
\[ m_{57} = m_{75} = -\frac{\beta}{3} I_4; \]
\[ m_{66} = I_2 + \frac{\left( a^2 + \beta^2 \right) I_4}{9} I_6; \]
\[ m_{67} = m_{76} = I_3 + \frac{\left( a^2 + \beta^2 \right) I_5}{9} I_6; \]
\[ m_{77} = I_4 + \frac{\left( a^2 + \beta^2 \right) I_6}{9} I_6. \]