Abstract—In this paper, the normalized integrals of mechanical energy of active and semi-active stiffness SDOF systems are minimized. The optimal active performance index is found analytically by the Linear Quadratic Regulator (LQR) controller. The gains of active control force are chosen from the limit of the ideal LQR controller. The minimum performance index is 0.5. We also prove that the semi-active system can reach the optimal active performance index of 0.5. The optimal semi-active stiffness control can approach the ideal active control in term of performance by fixing the damping ratio of 0.5 then switching stiffness from the large value to zero.

Keywords—Analytical optimization, bi-linear state space system, quadratic integral.

I. INTRODUCTION

The semi-active stiffness method control the device’s stiffness to affect the controlled system. Due to its performance close to fully active control system while the control algorithm is simple, the researches are actively conducted in implemented devices and control strategies, see for example the reviews [1-4] and references therein.

Besides the design of the semi-active device, many issues are involved in designing control laws including model, feedback architecture, performance objective, control methodology, stability and robustness. Many publications in literature propose, simulate and experimentally verify semi-active controls but they may fail when aiming at the optimal performance. The clear theoretical optimal solutions, therefore, are of great interest to evaluate a controller.

Under harmonic excitation, in [5-6], some ground hook controllers have been optimized, in [7, 8], the optimal controllers for some classes of on-off damping SDOF isolation was derived. In [9] the general controller containing skyhook, groundhook and bang-bang controller is optimized. However, to the best of our knowledge, there is still no closed form optimal controller for the system under free vibration.

This paper formulates the integral of mechanical energy of the fully active and semi-active stiffness SDOF systems. The formulation relies on the solutions to the Lyapunov equations. The truly minimum of integral of mechanical energy is obtained.

II. OPTIMIZATION PROBLEM FORMULATION

The semi-active stiffness and active controlled SDOF systems are shown in Fig. 1.

In Fig. 1a, the mass \(m\) is supported by the spring \(k\) and damper \(c\) and is free vibrated with the displacement \(z\). In Fig. 1b, the spring and damper are replaced by the active control force \(f\). For demonstration, the initial displacement condition in considered. The initial velocity problem should be studied in the future. The initial condition is taken as:

\[
z(0) = z_0, \quad \dot{z}(0) = 0
\]

The free vibrations of the systems in Fig. 1 are driven by the following classical equations:

\[
\begin{align*}
\ddot{z}(t) + 2\zeta \omega_n \dot{z}(t) + \omega_n^2 z(t) &= 0; & 0 \leq t < t_i \\
\ddot{z}(t) + 2\zeta \omega_n \dot{z}(t) + \gamma_i \omega_n^2 z(t) &= 0; & t_i \leq t
\end{align*}
\]

(Fig. 1a) (1)

\[
\ddot{z}(t) + \alpha_i \dot{z}(t) + \alpha_i z(t) = 0 \quad (\text{Fig. 1b})
\]

(2)

where: \(\omega_n = \sqrt{k_n/m}\) is the undamped nominal natural circular frequency, \(k_n\) is the nominal stiffness, \(\zeta\) is the passive damping ratio, the multiplier \(\gamma_i (i = 1, 2)\) multiplies with \(k_n\) to
introduce the bi-state stiffness. For the active system, in (2), \( \alpha_1 \) and \( \alpha_2 \) are two gains of the active control force.

The integral of the normalized mechanical energy is considered as the performance index:

\[
J = \frac{1}{\omega_0} \int_0^\infty \frac{\omega_0^2 z^2 + \dot{z}^2}{2z_0^2} dt
\]

(3)

The optimization problem of the active controlled system to find \( \alpha_1 \) and \( \alpha_2 \) to minimize the performance index (3). The corresponding problem of the semi-active stiffness controlled system is to find \( \alpha_1, \gamma_1, \gamma_2 \) and \( \alpha_1 \) to minimize the same index.

III. OPTIMAL ACTIVE CONTROL

The active control system (2) is written as:

\[
x = (A + B_0 G)x
\]

(4)

where \( x \) denote the state vector as:

\[
x = \frac{1}{z_0} \left[ z \quad \frac{z}{\omega_0} \right]^T
\]

with the initial condition

\[
x_0 = [1 \quad 0]^T
\]

(5)

The other matrices are follows:

\[
A = \begin{bmatrix} 0 & \omega_0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \quad 1 \end{bmatrix}, \quad G = \begin{bmatrix} \frac{\alpha_1}{\omega_0} & \alpha_2 \end{bmatrix}
\]

The active control force is:

\[
u = Gx
\]

The optimal active performance index is found analytically by the Linear Quadratic Regulator (LQR) controller [10]. In this case, the gains \( \alpha_1 \) and \( \alpha_2 \) in (2) are chosen from the limit of the ideal LQR controller.

The optimum gain matrix is found by minimize the quadratic performance index:

\[
J_{\text{LQR}} = \int_0^\infty \left( \frac{\alpha_1}{\omega_0} \frac{x^2 + \dot{x}^2}{2} + \frac{r}{\omega_0} u^2 \right) dt
\]

(6)

where \( r \) is a nondimensional weighting value of the control force in the LQR problem. In the limit case, when \( r \) tends to zero, we obtain the best active control. The performance index (6) is written in the quadratic integral as:

\[
J = \int_0^\infty \left( x^T Q x + \frac{r}{\omega_0} u^2 \right) dt
\]

where the weighting matrix \( Q \) is taken from

\[
Q = \frac{\omega_0}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

(7)

In the usual LQR control [10], the gain matrix \( G \) is defined by the expression:

\[
G = -\frac{\omega_0}{r} B^T S
\]

(8)

where the matrix \( S \) is defined by the algebraic Riccati equation:

\[
SA + A^T S - \frac{\omega_0}{r} SBB^T S + Q = 0
\]

(9)

Moreover, the resulting value of the performance index is:

\[
J_o = x_0^T S x_0
\]

(10)

Denote the components of the matrix \( S \) as:

\[
S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_3 \end{bmatrix}
\]

(11)

we write the Riccati equation (9) in the following form:

\[
\begin{bmatrix} S_1 & S_2 \\ S_2 & S_3 \end{bmatrix} \begin{bmatrix} 0 & \omega_0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_1 & 0 \\ \omega_0 & \frac{\omega_0}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0
\]

(12)

or

\[
\begin{bmatrix} 0 & S_2 \\ S_1 & 2S_2 \end{bmatrix} - \frac{1}{r} \begin{bmatrix} S_2 & S_1 \\ S_2 & S_3 + \frac{1}{2} \end{bmatrix} + \frac{1}{2} = 0
\]

(13)

which is written as 3 equations:

\[
\begin{align*}
-\frac{1}{r} S_2^2 + \frac{1}{2} &= 0 \\
S_1 - S_2 S_3 &= 0 \\
2 S_2 - \frac{1}{r} S_3^2 + \frac{1}{2} &= 0
\end{align*}
\]

(14)

The equations are solved as:

\[
\begin{align*}
S_2 &= \sqrt{\frac{r}{2}} \\
S_1 &= \frac{\sqrt{2r + \frac{1}{2}}}{r} \\
S_3 &= \frac{S_1 S_2}{r} \left( \sqrt{2r + \frac{1}{2}} \right)^2
\end{align*}
\]

(15)

Substituting (5) and (11) into (10), the performance index has form:

\[
J_o = x_0^T S x_0 = S_1
\]

(16)

The minimum active performance index is attained as \( r \) tends to zero. From (15), (16) we have:

\[
J_{\text{min}} = \lim_{r \to 0} S_1 = \frac{1}{2}
\]

In brief the minimum performance index of the active controlled system is 0.5.

IV. OPTIMAL SEMI-ACTIVE STIFFNESS CONTROL

In this section, we also prove that the semi-active stiffness control can give the performance achieving the best
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active one. The performance index (3) is written in the quadratic integral as:

$$ J = \int_0^\infty \dot{x}^T Q \dot{x} dt + \frac{1}{t} \int t \dot{x}^T Q \dot{x} dt $$  \hspace{1cm} (17) $$

where the weighting matrix is taken from (7). The equation (1) is written in the state space form:

$$ \begin{cases} \dot{x} = A_i x, & 0 < t < t_i \\ x = A_i x, & t_i < t \end{cases} $$  \hspace{1cm} (18) $$

where the state vector is (4) and the system matrices are:

$$ A_1 = \omega_j \begin{bmatrix} 0 & 1 \\ -\gamma_1 & -2\zeta \end{bmatrix} $$
$$ A_2 = \omega_j \begin{bmatrix} 0 & 1 \\ -\gamma_2 & -2\zeta \end{bmatrix} $$  \hspace{1cm} (19) $$

The solution of the bi-linear system (18) can be found as:

$$ x(t) = e^{A_i t} x_0, \hspace{1cm} 0 \leq t \leq t_i $$
$$ x(t) = e^{A(t-t_i)} e^{A_i t} x_0, \hspace{1cm} t_i \leq t $$  \hspace{1cm} (20) $$

where the matrix exponential is defined as usual [11]:

$$ e^{A_i t} = \sum_{j=0}^{\infty} \frac{A_i^j t^j}{j!} $$
$$ e^{A_i t} = \sum_{j=0}^{\infty} \frac{A_i^j t^j}{j!} $$  \hspace{1cm} (21) $$

The performance index (17) can be calculated without integration applying the algebraic Lyapunov matrix equation as

$$ J = x_0^T P_1 x_0 + x_0^T e^{A_i t} (P_2 - P_i) e^{A_i t} x_0 $$
$$ + \lim_{t \to \infty} x_0^T e^{A_i t} e^{{A(t-t_i)}} (-P_i) e^{A(t-t_i)} e^{A_i t} x_0 $$  \hspace{1cm} (22) $$

where: $P_i$ (i = 1, 2) are the symmetric matrices solved from the Lyapunov matrix equations [11]:

$$ P_i A_i + A_i^T P_i + Q = 0 $$
$$ P_i A_i + A_i^T P_i + Q = 0 $$  \hspace{1cm} (23) $$

Using the symbolic computation programs, such as MATLAB® Symbolic Math Toolbox, the matrix exponential (21) is determined by:

$$ e^{A_i t} = e^{-\zeta \gamma_i t} \begin{bmatrix} \cosh(\omega_j q_i t) & \sinh(\omega_j q_i t) \\ +\zeta \sinh(\omega_j q_i t) & q_i \end{bmatrix} \begin{bmatrix} \cosh(\omega_j q_i t) \\ -\gamma_i \sinh(\omega_j q_i t) \end{bmatrix} $$
$$ e^{A_i t} = e^{-\zeta \gamma_i t} \begin{bmatrix} \cosh(\omega_j q_i t) \sinh(\omega_j q_i t) \\ +\zeta \sinh(\omega_j q_i t) & q_i \end{bmatrix} \begin{bmatrix} \cosh(\omega_j q_i t) \\ -\gamma_i \sinh(\omega_j q_i t) \end{bmatrix} $$  \hspace{1cm} (24) $$

where

$$ q_i = \sqrt{\zeta^2 - \gamma_i}, \hspace{1cm} i = 1, 2 $$  \hspace{1cm} (25) $$

Note that:

$$ \lim_{t \to \infty} e^{-\zeta \gamma_i t} \cosh(\omega_j q_i t) $$
$$ \lim_{t \to \infty} e^{-\zeta \gamma_i t} \sinh(\omega_j q_i t) $$
$$ = \begin{cases} 1 & \zeta > \sqrt{\gamma_i} \\ 0 & \zeta \leq \sqrt{\gamma_i} \end{cases} \hspace{1cm} (26) $$

$$ \lim_{t \to \infty} e^{-\zeta \gamma_i t} \sinh(\omega_j q_i t) $$
$$ \lim_{t \to \infty} e^{-\zeta \gamma_i t} \cosh(\omega_j q_i t) $$
$$ = \begin{cases} 1 & \zeta < \sqrt{\gamma_i} \\ 0 & \zeta \geq \sqrt{\gamma_i} \end{cases} \hspace{1cm} (27) $$

The limits (26) and (27) reduces to:

$$ \lim_{t \to \infty} e^{A_i t} = 0 $$

where: $0$ is the 2x2 zero matrix. The performance index (22) then is reduced to:

$$ J = x_0^T P_1 x_0 + x_0^T e^{A_i t} (P_2 - P_i) e^{A_i t} x_0 $$  \hspace{1cm} (28) $$

Substituting (17) and (19) into the Lyapunov matrix equations (23) to solve

$$ P_i = \frac{1}{2} \begin{bmatrix} \zeta + 1 + \gamma_i & \frac{1}{\gamma_i} \\ \frac{1}{2\gamma_i} & \frac{1}{4\zeta} \left( \frac{1}{\gamma_i} + 1 \right) \end{bmatrix}, \hspace{1cm} i = 1, 2 $$

Substituting $P_1$ and $P_2$ from (29), $e^{A_i t}$ from (24) and $x_0$ from (5) into (28), some simplifications yield:

$$ J = \frac{1}{2} \begin{bmatrix} \zeta + 1 + \gamma_i & \frac{1}{4\zeta} \\ \frac{1}{2\gamma_i} & \frac{1}{4\zeta} \left( \frac{1}{\gamma_i} + 1 \right) \end{bmatrix} \begin{bmatrix} \cosh(\omega_j q_i t) \\ +\zeta \sinh(\omega_j q_i t) \end{bmatrix} \begin{bmatrix} 1 - \frac{\zeta}{4\zeta} \\ \gamma_i \gamma_i \end{bmatrix} $$
$$ e^{-\zeta \omega_i t} \gamma_2 - \gamma_1 $$
$$ + \frac{1}{\gamma_i \gamma_i} \begin{bmatrix} \frac{1}{\gamma_i} & \frac{1}{4\zeta} \left( \frac{1}{\gamma_i} + 1 \right) \end{bmatrix} \begin{bmatrix} \cosh(\omega_j q_i t) \\ +\zeta \sinh(\omega_j q_i t) \end{bmatrix} $$
$$ - \gamma_i^2 \sin^2(\omega_j q_i t) \frac{1}{4\zeta \gamma_i \gamma_i} $$  \hspace{1cm} (30) $$

The optimization problem is to find the switching time $t_i$, the passive damping $\zeta$ and the switching stiffness multiplier $\gamma_1$ and $\gamma_2$ to minimize $J$ in (30).
The optimal condition \( \frac{\partial J}{\partial t_1} = 0 \) can be simplified to:

\[
\cosh(\omega_0 \sqrt{s^2 - \gamma_1 t_{1m}^2}) + \left( \frac{\xi - \gamma_2 + 1}{2s} \right) \sinh(\omega_0 \sqrt{s^2 - \gamma_1 t_{1m}^2}) = 0
\]

which can be solved for the optimal switching time:

\[
t_{1m} = \frac{\omega_0 \sqrt{\gamma_1 - \xi^2}}{\arccos \left( 1 - \frac{2\xi^2}{(\gamma_2 + 1)/\gamma_1} \right)}
\]

(32)

Use (31) in (30), after some manipulations, we reduce the performance index to:

\[
J_e = \frac{\xi}{2\gamma_1} + \frac{1 + \gamma_1}{8\xi} + \frac{\gamma_2 - \gamma_1}{4\xi^2} e^{-2\omega_0 t_{1m}} \left( \left( \frac{\xi + \gamma_1}{4\xi^2} \right) \left( \gamma_2 + 1 \right)^2 + 1 + 2\gamma_2 \right)
\]

(33)

The plots of \( J_e \) versus \( \gamma_1 \) and \( \gamma_2 \) for various values of \( \zeta \) are shown in Fig.2, 3, 4

As seen in Figs. 2, 3, 4 the minimum of \( J_e \) is obtained when \( \gamma_1 \) is large (tends to infinity) and \( \gamma_2 \) is small (tends to zero). It is intuitive because in the first time interval \( (0 < t < t_{1m}) \), the stiffness should be as large as possible to pull the mass return to the equilibrium as quick as possible. In the second time interval, the stiffness should be small to prevent the overshoot. Therefore, first let \( \gamma_1 \) tends to infinity. Substitute (32) into (33), then take the limit \( \gamma_1 \rightarrow \infty \), after some manipulations we have

\[
\lim_{\gamma_1 \rightarrow \infty} J_e = \frac{(\gamma_2 + 1)^2 + 4\xi^2}{8\xi^2 \sqrt{\gamma_2 + 1}}
\]

(34)

The minization of (34) versus \( \gamma_2 \) and \( \zeta \) give:

\[
\begin{align*}
\gamma_2^{\text{opt}} &= 0 \\
\zeta^{\text{opt}} &= \frac{1}{2}
\end{align*}
\]

(35)

and the final value of the minimized performance index of the on-off stiffness control is:

\[
J_{e1}^{\text{opt}} = \frac{1}{2}
\]

Using (35) in (32) gives the switching time

\[
t_{1m}^{\text{opt}} = \frac{\arccos \left( 1 - \frac{2\gamma_2}{\gamma_1} \right)}{\omega_0 \sqrt{\gamma_1 - \frac{1}{4}}}
\]

In brief, the performance index of the semi-active stiffness control can approach the one of fully ideal LQR control by the following on-off stiffness strategy. First, the damping ratio \( \zeta \) is fixed at 0.5. Then a very large stiffness is applied. Denote \( \gamma_1 \) as the ratio between the very large stiffness applied and the uncontrolled nominal stiffness. At the time \( t_{1m}^{\text{opt}} = \arccos \left( 1 - \frac{1}{\gamma_1} \right) \left/ \omega_0 \sqrt{\gamma_1 - \frac{1}{2}} \right. \), the stiffness is turned to zero. It is clear that the above performance achieves the ideal active one (=0.5). However, the requirement of very large stiffness is a disadvantage.
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V. CONCLUSIONS

This paper considers the free vibration of an active or semi-active stiffness controlled system. The normalized integrals of mechanical energy of LQR active and semi-active stiffness SDOF systems are minimized. The main result of this paper is to solve analytically the minimization problem of the semi-active stiffness SDOF, although the performance index is a complex function of four variables. Both the active and semi-active stiffness controlled systems can reach the optimal performance index of 0.5. The optimal semi-active stiffness can approach the ideal active control in term of performance but the requirement of switching from a very large to zero stiffness is a disadvantage.

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