Exact receptance function of a cracked axially functionally graded beam

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Abstract—This paper presents the exact receptance function of a cracked axially functionally graded (AFG) beam. The receptance function of the cracked AFG beam is presented in the form of a power series. The influences of the cracks and the varying ratios of properties along the beam on the receptance function are investigated. The numerical calculations show that when there are cracks, the receptance function of beams has sharp changes at the crack positions. These sharp changes are amplified by using the receptance curvature that can be used for inspecting the crack existence visually. In this paper, the derivation of the exact receptance function is presented and numerical simulations are provided to justify the theory.

Keywords—Receptance; frequency response function; crack, functionally graded material beam.

I. INTRODUCTION (HEADING 1)


The vibration problem of cracked structural members has attracted the attention of many researchers since the cracks in components of a structure can potentially reduce its safety. However, there were only a few works considering the vibration problem of cracked FGM beams. Yang et al. [10] presented a theoretical study in free vibration of cracked FGM beams based on Bernoulli–Euler beam theory. Wei et al. [11] proposed a transfer matrix based method for solving the free vibration of cracked FGM beams with axial loading, rotary inertia and shear deformation. Aydin [12] studied free vibration of FGM beams containing arbitrary number of open edge cracks using the Kirchhoff–Love method. Rajasekaran et al. [13] presented a method for studying free vibration of bi-directional functionally graded single/multi-cracked beams using finite element beam model. Mao et al. [14] proposed an FEA method to study the vibration and wave propagation in FGM beams with multiple inclined cracks by introducing the local flexibility matrix caused by the cracks. However, these studies were often limited to FGM beams where the material properties vary along the height.

In this paper a method for determining the exact receptance function of an AFG beam with an arbitrary number of cracks is presented. The receptance function is found in the form of power series based on the Adomian decomposition. The coefficients of the power series are calculated by a recurrent formula. The derivation of the mode shape formula is given in detail and some numerical simulations are provided to demonstrate the theory.

II. THEORETICAL BACKGROUND

Let us consider the uniform AFG Euler-Bernoulli beam with length $L$ with an arbitrary number of cracks as shown in Fig. 1.

![Fig. 1. AFG Euler-Bernoulli beam](image-url)
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Using the notation \( \xi = \frac{z}{L} \) which is the non-dimensional coordinate, the receptance at \( \xi \) due to the force at \( \xi_f \) at the forcing frequency \( \omega \) can be obtained as [15]:

\[
\alpha(\xi, \xi_f, \omega) = \Phi^T(\xi)(K - \omega^2 M)^{-1}\Phi(\xi_f) \tag{1}
\]

where:

\[
M = \begin{bmatrix}
\int_0^\xi \rho \phi_i^2 d\xi & \int_0^\xi \rho \phi_i \phi_j d\xi & \ldots & \int_0^\xi \rho \phi_n^2 d\xi \\
\int_0^\xi \rho \phi_i \phi_j d\xi & \int_0^\xi \rho \phi_j^2 d\xi & \ldots & \int_0^\xi \rho \phi_n \phi_m d\xi \\
\ldots & \ldots & \ldots & \ldots \\
\int_0^\xi \rho \phi_i \phi_m d\xi & \int_0^\xi \rho \phi_m \phi_j d\xi & \ldots & \int_0^\xi \rho \phi_n^2 d\xi \\
\int_0^\xi \rho \phi_i^2 d\xi & \int_0^\xi \rho \phi_i \phi_j d\xi & \ldots & \int_0^\xi \rho \phi_n^2 d\xi
\end{bmatrix} \tag{2}
\]

\[
K = \frac{1}{EI} \begin{bmatrix}
\int_0^\xi E(\xi) \phi_i^2 d\xi & \int_0^\xi E(\xi) \phi_i \phi_j d\xi & \ldots & \int_0^\xi E(\xi) \phi_n^2 d\xi \\
\int_0^\xi E(\xi) \phi_i \phi_j d\xi & \int_0^\xi E(\xi) \phi_j^2 d\xi & \ldots & \int_0^\xi E(\xi) \phi_n \phi_m d\xi \\
\ldots & \ldots & \ldots & \ldots \\
\int_0^\xi E(\xi) \phi_i \phi_m d\xi & \int_0^\xi E(\xi) \phi_m \phi_j d\xi & \ldots & \int_0^\xi E(\xi) \phi_n^2 d\xi \\
\int_0^\xi E(\xi) \phi_i^2 d\xi & \int_0^\xi E(\xi) \phi_i \phi_j d\xi & \ldots & \int_0^\xi E(\xi) \phi_n^2 d\xi
\end{bmatrix} \tag{3}
\]

\[
\Phi(\xi) = [\phi_1(\xi), \phi_2(\xi), \ldots, \phi_n(\xi)]^T \tag{4}
\]

Here: \( \omega \) and \( \phi_i(\xi) \) are the \( i^{th} \) natural frequency mode shape at position \( \xi \), respectively. Assume that the elasticity modulus \( E(x) \) and the mass density \( \rho(x) \) of the intact AFG Euler-Bernoulli beam are varied as follows [16]:

\[
E(\xi) = E_0 \left[ 1 - \alpha_1 \xi^{m_1} \right], \quad \rho(\xi) = \rho_0 \left[ 1 - \alpha_2 \xi^{m_2} \right] \tag{5}
\]

In order to derive the mode shape of a cracked AFG beam, the governing equation of bending vibration of the beam is applied as follows:

\[
\left[ E(\xi) I \phi_i''(\xi) \right]' - \omega^2 \mu(\xi) \phi_i(\xi) = 0 \tag{6}
\]

where: \( E_0 \) and \( \rho_0 \) are Young’s modulus and mass density at \( \xi = 0 \), respectively; \( 0 < \alpha_1 < 1; n \) is the varying ratio of material properties. The flexural stiffness \( E(\xi)I \) of the cracked beam can be described as follows [17]:

\[
E_c(\xi) = E_0 \left[ 1 - \alpha_1 \xi^{m_1} \right] \left[ 1 - \sum_{i=1}^p \gamma_i \delta(\xi - \xi_{i_o}) \right] \tag{7}
\]

where: \( p \) singularities, given by Dirac’s deltas centered at abscissa \( \xi_{i_o} \), \( i = 1, \ldots, p \) represent \( p \) concentrated cracks. The parameters \( \gamma_i \) are parameters representative of the crack intensity.

Substituting Eq. (7) into Eq. (6) and simplifying, yields:

\[
\phi_i'' + \left[ \frac{2n\alpha_1 \xi^{m_1}}{1 - (\beta \xi)^m} \phi_i'' + \frac{n(n-1)\alpha_2 \xi^{m_2}}{1 - (\beta \xi)^n} \phi_i'' + \frac{\gamma_i \alpha \xi^{m_1}}{1 - (\beta \xi)^m} \phi_i \right] \tag{8}
\]

where \( \beta = \sqrt[2]{\alpha_1} \), \( \lambda = \frac{\alpha_2 \lambda_1 L}{E_0 I} \), \( \gamma_i = \frac{1}{1 - \sum_{i=1}^p \gamma_i \delta(\xi - \xi_{i_o})} \)

Eq. (8) can be solved by the Adomian decomposition method in which \( \phi \) is decomposed into the infinite sum of convergent series:

\[
\phi_i(\xi) = \sum_{k=0}^{\infty} C_k \xi^k \tag{9}
\]

Introducing linear operators where \( \xi = \frac{d^4}{d\xi^4} \) and \( \xi^{-1} = \int_0^\xi \int_0^\xi \int_0^\xi \int_0^\xi (\ldots) d\xi d\xi d\xi d\xi \), we have:

\[
\xi^{-1} \phi_i(\xi) = \xi^{-1} \left[ \phi_i(\xi) \right] = \phi_i - C_0 - C_1 \xi - C_2 \xi^2 - C_3 \xi^3 \tag{10}
\]

Applying these linear operators for Eq. (8) one can derive:

\[
\phi_i = \sum_{k=0}^{\infty} C_k \xi^k + \sum_{k=0}^{\infty} n(n-1)\alpha_2 \sum_{i=0}^{\infty} \frac{\beta_i^m (i+1)(i+2)}{(k+n-1)(k+n+1)(k+n+2)} C_{k-m+2} \\
+ \sum_{k=0}^{\infty} \frac{2n\alpha_1 \xi^{m_1} \phi_i''}{(k+n)(k+n+1)(k+n+2)} C_{k-m+3} + \sum_{k=0}^{\infty} \frac{\gamma_i \alpha \xi^{m_1} \phi_i}{(k+1)(k+2)(k+3)(k+4)} C_{k-m+4} \\
- \sum_{k=0}^{\infty} \frac{\lambda^m \alpha \xi^{m_1} \phi_i}{(k+n+1)(k+n+2)(k+n+3)(k+n+4)} \tag{11}
\]

The coefficients \( C_k \) where \( k < 4 \) can be determined from the boundary conditions. Let us consider a simply supported beam, the boundary conditions can be expressed as:

\[
\phi(0) = 0, \quad \phi'(0) = 0 \tag{12}
\]

\[
\phi(1) = 0, \quad \phi'(1) = 0 \tag{13}
\]

From Eqs. (14) and (15) we have:

\[
C_0 = 0, C_1 \neq 0, C_2 = 0, C_3 \neq 0 \tag{14}
\]

For \( k \geq 4 \) the coefficients \( C_k \) can be calculated from the recurrent relations depending on the value of \( n \) as follows:
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\[ C_i = \frac{n(n-1)\alpha_i}{(k-3)(k-2)(k-1)k} \sum_{i=0}^{n} \beta^{m_i} (i+1)(i+2)C_{m+p_i+2} \]

\[ + \frac{2ndt}{(k-3)(k-2)(k-1)k} \sum_{i=0}^{n} \beta^{m_i} (i+1)(i+2)(i+3)C_{m+p_i+3} \]

\[ + \frac{\gamma_i\lambda^2}{(k-3)(k-2)(k-1)k} \sum_{i=0}^{n} \beta^{m_i} \gamma_i C_{m+p_i} \]

where: \( m_1 = k - n - 3 \) \( \frac{n}{n} \) \( m_2 = k - n - 2 \)

\[
\begin{align*}
\gamma_i & = \frac{x_i}{20018h_i - 1}, \\
\gamma_i & = \frac{h}{L} C(\beta)
\end{align*}
\]

Here: \( C(\beta) = \frac{\beta(2-\beta)}{0.9(\beta-1)} \)

Although there are four cracks with depths up to 30%, the influence of the cracks on the receptance of the beam cannot be seen visually from Fig. 2. This can be explained by the fact that, the influence of the crack on the mode shape is significant only when the crack depth is large. Therefore, in order to investigate the effect of cracks on the receptance of beam, the receptance curvature matrix which is derived by taking the second derivative of the receptance will be applied.

III. NUMERICAL SIMULATION

In this paper, numerical simulations of a simply supported beam with four cracks at \( 0.34L \), \( 0.5L \), \( 0.7L \) and \( 0.8L \) is presented. The beam consists of two constituent materials of steel and alumina (Al\(_2\)O\(_3\)). The material properties are \( E_0 = 3.9 \times 10^{11} \) N/m\(^2\) and \( \rho_0 = 3960 \) kg/m\(^3\) for alumina and \( E_0 = 2.0 \times 10^{11} \) N/m\(^2\), \( \rho_1 = 7800 \) kg/m\(^3\) for steel; \( L = 1 \) m; \( b = 0.02 \) m; \( h = 0.02 \) m. The receptance matrices are calculated at 50 points spaced equally on the beam while the force moves along these points. The crack intensity \( \gamma \) in Eq. (7) can be calculated from the following formula [18].

The \( i^{th} \) mode shape \( \phi_i \) corresponding to \( \lambda_i \) is calculated from Eq. (12). Once the mode shape \( \phi_i \) is determined, matrices \( M \) and \( K \) will be obtained, thus the receptance formula (1) is determined.
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Fig. 2. Receptance matrices of cracked beam, crack depth = 30%

Figs. 3 and 4 present the receptance curvature matrices of the beam with the crack depths ranging from 10% to 30%. In these figures, graphs a), c), e) are receptance curvature matrices and graphs b), d), f) are receptance curvature extracted from the receptance curvature matrices at the fixed position of 0.3L. As can be observed from these figures, there are sharp peaks in the receptance curvature matrices as well as in the receptance curvatures extracted at 0.3L. The positions of the sharp peaks coincide with the positions of the cracks. When the crack depth increase, the heights of sharp peaks increase. These results mean that the cracks affect the receptance curvatures by producing sharp peaks at the crack positions in the receptance curvatures.

It is noted that when the cracks are located at the peaks of mode shapes the sharp peaks caused by cracks are larger than the case where the cracks are located far from the peaks of mode shapes. Meanwhile, the sharp peaks caused by cracks are smallest when the cracks are located at node points of mode shapes. The simulation results show that the receptance function of beam can be applied for crack detection by inspecting the sharp peaks in the receptance curvatures. The sharp peaks imply the crack existence and the positions of the peaks show the positions of cracks.
Figure 3: Receptance curvature, crack depth = 10%

Figure 4: Receptance curvature, crack depth = 20%
IV. CONCLUSION

In this paper, the general form of receptance function of the cracked AFG beam is presented. The derived receptance function can be applied easily to estimate the dynamic response of a cracked beam at any point to a harmonic force applied at any point. The graphs of the receptance matrices of the cracked AFG beam at fundamental frequencies are presented. The effect of cracks on the receptance is investigated.

When the excitation frequency is equal to fundamental frequencies of beam, there are maximum points in the receptance matrices which coincide with the maximum points of the corresponding mode shapes. There are also node points in the receptance matrices that coincide with the node points of the corresponding mode shape. The effect of cracks on these receptance matrices is very small that cannot be inspected visually. However, the effect of cracks on the receptance matrices can be amplified by using the receptance curvatures. When there are cracks, there exists significant sharp peaks in the receptance curvatures at crack positions. The heights of sharp peaks increase when the depths of cracks increase.

The proposed method can be applied for crack detection. The sharp peaks in the receptance curvatures can be used as an indication of the crack existence and the heights of the sharp peaks can be used to evaluate the sizes of cracks.
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REFERENCES