Upper and lower iteration boundaries on longitudinal-transverse elastic modulus of transversely isotropic unidirectional composites

Hanh Vuong  
Institute of Mechanics  
(Vietnam Academy of Science and Technology)  
Hanoi, Vietnam  
vuongmyhanh.imech@gmail.com

Chinh Pham  
Institute of Mechanics  
(Vietnam Academy of Science and Technology)  
Hanoi, Vietnam  
pdchinh@imech.vast.vn

Abstract—Currently, fiber-reinforced materials have been widely applied in all fields such as science and technology as well as human life. Therefore, building a relationship between the effective properties of composite materials and those of the constituent components is of great scientific significance. Besides, it helps us to manufacture materials to meet practical requirements. In recent reports, the authors have built expressions closely related between the effective axial elastic modulus and the micro properties of the general n-component transversely isotropic materials. In particular, consider specifically with two axial elastic coefficients, Young's modulus and Poisson's ratio. Simultaneously calculating numerically and comparing with well-known bounds such as Hill-Paul, Hashin-Shtrikman, get perfectly reasonable results. Hence, this report continues to improve the above expressions, and at the same time we build new expressions for all 6 effective longitudinal-transverse elastic constants. Numerical illustrations and comparisons for real composite materials with a positive Poisson’s ratio are also considered.

Keywords—Longitudinal-transverse elastic modulus, transverse-isotropic unidirectional composites, Young's modulus, Poisson’s ratio, iteration bounds, multi-component material.

I. INTRODUCTION

Starting from the minimum energy and compliance minimum energy principles, combined with constant stress and strain test fields, variational boundaries for macroscopic elastic constants of composites have been constructed. The simplest one is Hill-Paul bound [1, 2], including arithmetic average (upper bound Voigt) and harmonic average (lower bound Reuss). This bound is still widely used in engineering calculations. Then, Hashin-Shtrikman [3-6] constructed the general possible polarization field to lead to the better bounds.

Separate and improved variational boundaries for different composite classes are subject of much literature, including sub-class of symmetric materials (unit cells) with additional correlation information of the composite microstructure [7, 8], boundaries for isotropic composites [9-11]. In particular, we are especially interested in the actual material class of transversely-isotropic fiber composites.

However, there are very few published direct boundaries for Young's elastic modulus and Poisson's ratio. In recent studies [12-15], we have established general boundary expressions for these two axial elastic coefficients. Simultaneously numerical illustrating and comparing with Hill-Paul and Hashin-Shtrikman ones, get perfectly reasonable results.

Inheriting and developing existing results, this report has the following main objectives:

Firstly, this report aims to further improve our previously constructed expressions to achieve better (tighter) bounds.

Secondly, we also establish new iterative expressions for all six effective longitudinal-transverse elastic coefficients (section IV). These expressions are new and have not been built in our previous reports.

Besides, section V applies these developed expressions to calculate real 2- and 3-component composites. Numerical results and comparisons with previous ones are also presented.

II. SCIENTIFIC BASIS

This section outlines the mechanical expressions of the problem, and classic boundaries are also presented. These classic boundaries are the basis for comparison with the new estimations of our reports.

A. Basic expressions

Consider $V$ as a representative volume element of the transversely isotropic unidirectional composites, consisting $n$ components $V_{a}$ with volume $v_{a}(\alpha = 1,...,n)$ respectively. Without loss of generality and for simplicity, we take $V = 1$ volume unit. Macroscopic elastic coefficients of composite
and microscopic elastic coefficients of material component $\alpha$ are denoted respectively as follows: $C_{\alpha}^{\text{eff}}, C_{\alpha}, S_{\alpha}^{\text{eff}}, S_{\alpha}$ are the stiffness and compliance elastic modulus, $K_{\alpha}^{\text{eff}}, K_{\alpha}$ are the transverse bulk deformation modulus, $\mu_{\alpha}^{\text{eff}}, \mu_{\alpha}$ are the transverse shear modulus, $\mu_{\alpha}^{\text{eff}}, \mu_{\alpha}$ are the longitudinal shear modulus, $E_{\alpha}^{\text{eff}}, E_{\alpha}$ are the longitudinal Young's modulus, $\nu_{\alpha}^{\text{eff}}, \nu_{\alpha}$ are the longitudinal Poisson's ratio, $\langle i \rangle = \int_V (i) dV$ is the volume averaging on $V$.

To determine the macroscopic elastic coefficients of the general composites, we use minimum energy and compliance minimum energy principles:

$$\sigma^0 : S^{\text{eff}} : \sigma^0 = \inf_{\{\sigma\} = \sigma^0} \int_V \sigma : S : \sigma d\mathbf{x} = \inf_{\{\sigma\} = \sigma^0} \langle \sigma : \varepsilon \rangle$$  \hspace{1cm} (1)

$$\varepsilon^0 : C^{\text{eff}} : \varepsilon^0 = \inf_{\{\varepsilon\} = \varepsilon^0} \int_V \varepsilon : C : \varepsilon d\mathbf{x} = \inf_{\{\varepsilon\} = \varepsilon^0} \langle \sigma : \varepsilon \rangle$$  \hspace{1cm} (2)

where $\varepsilon^0, \sigma^0$ are the constant strain and stress. The strain field $\varepsilon$ in (1) satisfies the compatibility equation, the stress field $\sigma$ in (2) satisfies the equilibrium condition.

From the general stress-strain relationship, we have longitudinal stress-strain relationships for transversely isotropic composites [7, 8]:

$$\sigma_{11} + \sigma_{22} = 2K \left( \varepsilon_{11} + \varepsilon_{22} \right) + 4\nu_{K} \varepsilon_{33}, \nu_{K} = vK;$$

$$\sigma_{33} = E_{K} \varepsilon_{33} + 2\nu_{K} \left( \varepsilon_{11} + \varepsilon_{22} \right), E_{K} = E + 4Kv^{2};$$

$$\varepsilon_{33} = \frac{1}{E} \left( \sigma_{11} + \sigma_{22} \right) - \nu_{K} \sigma_{33},$$

$$\varepsilon_{11} + \varepsilon_{22} = \frac{1}{2K} \left( \sigma_{11} + \sigma_{22} \right) - 2\nu_{K} \varepsilon_{33},$$

$$K_{E} = \left( \frac{1}{K} + \frac{4v^{2}}{E} \right)^{-1}.$$

In (3), besides the usual symbols $E, v, K$ used for isotropic composites, the report introduces additional longitudinal-transverse elastic coefficients as $v_{K}, E_{K}, v_{E}, K_{E}$. The elastic coefficients $K, E, v, K_{a}, E_{a}, v_{a}, K_{E}, E_{E}, K_{E}$ correspond to $K_{a}, E_{a}, v_{a}, K_{a}, E_{a}, v_{a}, K_{a}$ if the field $\sigma_{ij}(x), \varepsilon_{ij}(x)$ are given in phase $\alpha (x \in V \subset V)$, and correspond to $K_{\alpha}^{\text{eff}}, E_{\alpha}^{\text{eff}}, v_{\alpha}^{\text{eff}}, K_{\alpha}^{\text{eff}}, E_{\alpha}^{\text{eff}}, v_{\alpha}^{\text{eff}}, K_{\alpha}^{\text{eff}}$ if the volume average $\langle \sigma_{ij} \rangle, \langle \varepsilon_{ij} \rangle$ take the phase $\sigma_{ij}, \varepsilon_{ij}$. Although there exist relationship combining the exact values of these six longitudinal-transverse elastic moduluses, their best possible bounds generally should be derived as directly as possible.

B. Previous bounds

- Voigt-Ruess bounds:

With constant stress-strain trial fields, Hill-Paul [1, 2] have built the bounds on macroscopic elastic modulus for the transversely isotropic composites. Specifically with $K_{\text{eff}}, \mu_{\text{eff}}, \mu_{\text{eff}}$ we have upper bound Voigt $K_{V}, \mu_{V}, \tilde{\mu}_{V}$ and lower bound Reuss $K_{R}, \mu_{R}, \tilde{\mu}_{R}$ respectively:

$$K_{R} = \left( \sum_{a=1}^{n} v_{a} K_{a}^{-1} \right)^{-1} \leq K_{\text{eff}} \leq K_{V} = \sum_{a=1}^{n} v_{a} K_{a},$$

$$\mu_{R} = \left( \sum_{a=1}^{n} v_{a} \mu_{a}^{-1} \right)^{-1} \leq \mu_{\text{eff}} \leq \mu_{V} = \sum_{a=1}^{n} v_{a} \mu_{a},$$

$$\tilde{\mu}_{R} = \left( \sum_{a=1}^{n} v_{a} \tilde{\mu}_{a}^{-1} \right)^{-1} \leq \tilde{\mu}_{\text{eff}} \leq \tilde{\mu}_{V} = \sum_{a=1}^{n} v_{a} \tilde{\mu}_{a}.$$

- Hashin-Shtrikman bounds:

Besides, with the additional assumption of statistical isotropy in the transverse plane, Hashin-Shtrikman [3, 4] have constructed tighter boundaries for the transverse bulk deformation modulus $K_{\text{eff}}$:

$$K_{U}^{\text{HS}} \leq K_{\text{eff}} \leq K_{L}^{\text{HS}} = \left( \sum_{a=1}^{n} v_{a} K_{a} + \mu_{M} \right)^{-1} - \mu_{M},$$

$$\mu_{m} = \min \{ \mu_{1}, ..., \mu_{n} \}, \mu_{M} = \max \{ \mu_{1}, ..., \mu_{n} \}.$$

where $K_{U}^{\text{HS}}, K_{L}^{\text{HS}}$ are the upper and lower bounds of the Hashin-Shtrikman for $K_{\text{eff}}$.

III. PRIMARY ITERATION BOUNDS

This section summarizes the results that the authors have established in previous reports [12-15]. These expressions are called the primary iteration bounds, which are the basis for comparison with the new improved iteration bounds presented in the next section.

A. General boundary expressions

By selecting possible strain and stress fields, our previous reports have established following general upper and lower boundary expressions.

- General upper boundary expressions:

$$E_{V}^{\text{eff}} + 4K_{V} v_{\alpha}^{3} \leq E_{V}^{\text{eff}} \leq E_{V}^{\text{eff}} = \sum_{a=1}^{n} v_{a} \left( E_{a} + 4K_{a} v_{a}^{3} \right).$$

$$K_{V} \left( t + 2v_{a}^{3} \right)^{2} + E_{V}^{\text{eff}} \leq \sum_{a=1}^{n} v_{a} \left[ K_{a} \left( t + 2v_{a}^{3} \right) + E_{a} \right].$$

where $t$ is the free relation parameter between transverse bulk deformation and axial bulk deformation.

- General lower boundary expressions:
Upper and lower iteration boundaries on longitudinal-transverse elastic modulus of transversely isotropic unidirectional composites

\[
\frac{(2\tau - 1)^2}{E^\text{eff}} + \frac{1}{K^\text{eff}} \leq \frac{(2\tau \nu - 1)^2}{E_\nu} + \frac{\tau^2}{K_R},
\]

\[
E^\text{eff} \geq E_\nu = \sum_{\alpha = 1}^n v_\alpha E_\alpha, \quad E_\nu = \sum_{\alpha = 1}^n v_\alpha E_\alpha.
\]

where \( \tau \) is the free relation parameter, connecting the axial and transverse bulk stress.

B. Primary iteration bounds

Using the Lagrange multiplier method, optimizing the general boundaries (6, 7), combining the starting expressions (1-5), our previous reports have led to the primary iteration boundary expressions. These bounds have been only constructed for two macroscopic longitudinal elastic modulus of the composites. To distinguish these old iteration bounds from the new ones (that will be built in the following section), we add the symbol "o" to these primary ones.

- Boundary series for Poisson's coefficients:

\[
v_{1o}^L < v_{i+1}^L \leq v^\text{eff} \leq v_{1o}^L \leq v_{1o}^L, \quad v_{i+1}^L = \max \left\{ v_{1o}^L, v_{2o}^L \right\}, \quad v_{1o}^L = \min \left\{ v_{1o}^L, v_{2o}^L, v_{3o}^L, v_{4o}^L \right\}, \quad v_{i+1}^L = \frac{E_i F_{i+1}^b}{E_i F_{i+1}^b}, \quad F_{i+1}^d \geq 0, \quad E_i F_{i+1}^b, \quad F_{i+1}^d \geq 0, \quad \left( K^L \right)^{-1} F_{i+1}^d, \quad F_{i+1}^d < 0,
\]

- Initial value on \( E^\text{eff} \)

Then, by eliminating a positive term \( 4K^\text{eff} \left( v^\text{eff} \right)^2 \) to strengthen above inequality (10), we obtain an explicit initial upper bound on \( E^\text{eff} \):

\[
E^\text{eff} \leq E_\nu + 4\sum_{\alpha = 1}^n v_\alpha K_\alpha v_\alpha^2 = E_{KV} = E_{(1)},
\]

If a non-vanishing positive lower bound \( (K^2)_(-) \) exits, a smaller upper bound on \( E^\text{eff} \) can be deduced:

\[
E^\text{eff} \leq E_\nu + 4\sum_{\alpha = 1}^n v_\alpha K_\alpha v_\alpha^2 - 4(K^2)_{(-)}
\]

Unspecified term \( (K^2)_{(-)} \) should be as large as possible for the smallest possible bound \( E_{(1)} \) on \( E^\text{eff} \).

- Initial value on \( E^\text{eff} \)

The monotonic and bounded series \( E^U, v^U, v^L \) converge to the corresponding determined values.
Similarly to the derivation of inequality (12), (13), with the help of (10), (11), we derive a lower bound on $E_{K}^{\text{eff}}$:

$$E_{K}^{\text{eff}} \geq E_{V} + 4(K\nu^2)_{(-)} = E_{K}^{L(-)}.$$  \hspace{1cm} (14)

If $(K\nu^2)_{(-)}$ the bigger $E_{K}^{L(-)}$ the smaller (the better). At first, we eliminate $(K\nu^2)_{(-)}$ to strengthen above inequality, and deduce initial bound:

$$E_{K}^{\text{eff}} \geq E_{V} = E_{K}^{L(0)}.$$  \hspace{1cm} (15)

- Initial value on $K_{E}^{\text{eff}}$

From primary bounds (6)-(9) and lower bound on $K_{E}^{\text{eff}}$ (11), if $(\nu^2/E)_{(-)}$ is a lower bound on $(\nu^2/E)^2$, we have:

$$K_{E}^{\text{eff}} \leq \left[ \frac{1}{K_{E}^{U}} + \frac{4\nu^2}{E} \right]^{-1} = K_{E}^{U(+)}.$$  \hspace{1cm} (16)

Firstly, we eliminate positive term $(\nu^2/E)_{(-)}$ to strengthen above inequality, and deduce initial bound:

$$K_{E}^{\text{eff}} \leq K_{1}^{U} = K_{E}^{U(0)}.$$  \hspace{1cm} (17)

- Initial value on $\nu_{K}, \nu_{E}, \nu_{K}^{L}, \nu_{E}^{L}$

With definition $E_{K} = E + 4\nu^2 K$, we have

$$\nu^2 = \frac{1}{4} \left( \frac{E_{K} - E}{K} \right),$$

$$\nu_{K}^2 = (K\nu)^2 = \frac{1}{4} K \left( E_{K} - E \right).$$

Then, with the help of bounds (10), (11) and similar arguments we can deduce:

$$\left( \nu_{K}^{\text{eff}} \right)^2 \leq \frac{1}{4} \left( E_{K} - E \right) = \left( \nu_{U}^{L(+)} \right)^2,$$

$$\left( \nu_{E}^{\text{eff}} \right)^2 \leq \frac{1}{4} \frac{E_{K} - E}{K^{L}}, \left( \nu_{E}^{L(+)} \right)^2,$$

$$\nu_{K}^{\text{eff}} \leq \left( K^{L(-)} \nu_{K}^{U(+)} \right)^{\nu_{U}^{L(0)}}, \nu_{K}^{U(0)} \geq 0,$$

$$\nu_{K}^{\text{eff}} \leq \left( K^{L(-)} \nu_{K}^{U(0)} \right)^{\nu_{U}^{L(0)}}, \nu_{K}^{U(0)} \geq 0.$$

$$\nu_{E}^{\text{eff}} \leq \left( E_{K}^{U(+)} \nu_{E}^{L(+)} \right)^{\nu_{U}^{L(0)}}, \nu_{E}^{L(+)} \geq 0,$$

$$\nu_{E}^{\text{eff}} \leq \left( E_{K}^{L(-)} \nu_{E}^{L(0)} \right)^{\nu_{U}^{L(0)}}, \nu_{E}^{L(0)} \geq 0.$$

$$\left( \nu_{K}^{\text{eff}} \right)^2 \leq \frac{1}{4} \left( E_{K} - E \right) = \left( \nu_{K}^{U(+)} \right)^2,$$

$$\left( \nu_{E}^{\text{eff}} \right)^2 \leq \frac{1}{4} \frac{E_{K} - E}{K^{L}}, \left( \nu_{E}^{L(+)} \right)^2.$$  \hspace{1cm} (18)

where $E_{K}^{U(+)}, K_{E}^{L(-)}, K_{E}^{U(0)}$ are unspecific bound on $E_{K}^{U}, K_{E}^{L(-)}, K_{E}^{U(0)}$, respectively. Then we take $E_{K}^{L(0)} = E_{K}^{L(-)} = E_{K}^{L(0)}, K_{E}^{U(+)} = K_{E}^{U(0)}, K_{E}^{L(-)} = E_{K}^{L(0)}$, with almost similar arguments, from (10), (11) and primary bounds we can obtain new initial bounds:

$$v_{L(0)}^{U} \leq v_{L(0)}^{U}, v_{L(0)}^{L(0)}, v_{L(0)}^{U}, v_{L(0)}^{U} \leq v_{L(0)}^{U}, v_{L(0)}^{U} \leq v_{L(0)}^{U}, v_{L(0)}^{U} \leq v_{L(0)}^{U}.$$  \hspace{1cm} (19)

$$v_{L(0)}^{U} = \max \left\{ -v_{L(0)}^{U}, -v_{L(0)}^{U}, -v_{L(0)}^{U}, -v_{L(0)}^{U} \right\},$$

$$v_{L(0)}^{U} = \min \left\{ v_{L(0)}^{U}, v_{L(0)}^{U}, v_{L(0)}^{U}, v_{L(0)}^{U} \right\},$$

$$v_{L(0)}^{U} = \max \left\{ -v_{L(0)}^{U}, -v_{L(0)}^{U}, -v_{L(0)}^{U}, -v_{L(0)}^{U} \right\},$$

$$v_{L(0)}^{U} = \min \left\{ v_{L(0)}^{U}, v_{L(0)}^{U}, v_{L(0)}^{U}, v_{L(0)}^{U} \right\},$$

$$v_{L(0)}^{U} = \max \left\{ -v_{L(0)}^{U}, -v_{L(0)}^{U}, -v_{L(0)}^{U}, -v_{L(0)}^{U} \right\},$$

$$v_{L(0)}^{U} = \min \left\{ v_{L(0)}^{U}, v_{L(0)}^{U}, v_{L(0)}^{U}, v_{L(0)}^{U} \right\},$$

$$v_{L(0)}^{U} = \max \left\{ -v_{L(0)}^{U}, -v_{L(0)}^{U}, -v_{L(0)}^{U}, -v_{L(0)}^{U} \right\},$$

$$v_{L(0)}^{U} = \min \left\{ v_{L(0)}^{U}, v_{L(0)}^{U}, v_{L(0)}^{U}, v_{L(0)}^{U} \right\}.$$  \hspace{1cm} (20)
Upper and lower iteration boundaries on longitudinal-transverse elastic modulus of transversely isotropic composites

\[ v_{E(0)}^U = \min \left\{ v_{E1}^U, v_{E2}^U, v_{E3}^U, v_{E4}^U \right\}, \]
\[ v_{E(0)}^L = \max \left\{ -v_{E1}^U, -v_{E2}^U, -v_{E3}^U, -v_{E4}^U \right\}, \]
\[ v_{U2}^{(0)} = \frac{1}{2} E_{KV} \left( \frac{1}{K^L} - \frac{1}{K^U} \right)^{1/2}, \]
\[ v_{K(0)}^{(0)} = \frac{1}{2} K^U \left( \frac{1}{K^L} - \frac{1}{K^U} \right)^{1/2}, \]
\[ v_{L2}^{(0)} = \sum_{a=1}^n \alpha_a K_a v_a - \frac{1}{2} \left( K_v - K^L \right)^{1/2} \left( E_{KV} - E_v \right)^{1/2}, \]
\[ v_{U4}^{(0)} = \frac{v_v + 1}{E_v} \left( \frac{1}{K^L} - \frac{1}{K^U} \right)^{1/2} \left( 1 - \frac{1}{E_v} \right)^{1/2}, \]
\[ v_{L4}^{(0)} = \frac{E_{vK} v_{U2}^{(0)} - v_{U4}^{(0)} > 0}{\alpha_a K_a v_a - \frac{1}{2} \left( K_v - K^L \right)^{1/2} \left( E_{KV} - E_v \right)^{1/2}}, \]
\[ v_{U3}^{(0)} = \frac{E_{vK} v_{U2}^{(0)} - v_{U4}^{(0)} < 0}{\alpha_a K_a v_a - \frac{1}{2} \left( K_v - K^L \right)^{1/2} \left( E_{KV} - E_v \right)^{1/2}}, \]
\[ v_{K2}^{(0)} = \frac{E_{vK} v_{L2}^{(0)} - v_{L4}^{(0)} > 0}{\alpha_a K_a v_a - \frac{1}{2} \left( K_v - K^L \right)^{1/2} \left( E_{KV} - E_v \right)^{1/2}}, \]
\[ v_{L4}^{(0)} = \max \left\{ \alpha_a K_a v_a - \frac{1}{2} \left( K_v - K^L \right)^{1/2} \left( E_{KV} - E_v \right)^{1/2} \right\}. \]

(19)

Thus, the report has built initial boundary expressions for all 6 longitudinal-transverse elastic coefficients. We see that some original values in (12)-(19) are different (improved) from the corresponding values in the primary bounds [14, 15]. From here, we also get the new and improved iteration for two macroscopic longitudinal elastic modulus.

B. New iteration bounds

We consider the composite with positive Poisson’s ratio, so \( v_{E(0)}^U > 0 \). We easily see that if the positive lower bounds \((K_v^2)_{(0)}\) and \((v^2/E)_{(0)}\) do not disappear, and they satisfy the conditions in (1), (2), (10)-(19), hence we can improve the primary bounds (6)-(9) further in the following iteration procedure. This process is similar to and builds on the arguments in our previous reports [12-15].

We start with initial values in (12)-(19). Select \((K_v^2)_{(0)} = (K_{vL})^2_{(0)}\), \((v^2/E)_{(0)} = \left( \frac{v^2}{E} \right)_{(0)}\), replace in those initial value expressions to get values at step \((i = 1)\).

If \( E_{K(i)}, v_{K(i)}^{L}, K_{E(i)}, v_{E(i)}^{L}, v_{E(i)}^{L}, v_{E(i)}^{L} \) have been available after step \((i)\), we move to the step \((i+1)\), with the help of (10)-(19) we obtain:

\[ E_{K(i+1)}^{eff} \leq E_{K(i+1)}^{eff} = E_{KV} - 4v_{K(i+1)}^{U} < E_{K(i)}, \]
\[ E_{K(i+1)}^{eff} \geq E_{K(i+1)}^{eff} = E_{v} + 4v_{K(i+1)}^{L} > E_{K(i)}, \]
\[ K_{K(i+1)}^{eff} \leq K_{K(i+1)}^{eff} = \left( \frac{1}{K^L} \right)^{-1} + 4v_{K(i+1)}^{U} < K_{K(i+1)}, \]
\[ v_{K(i+1)}^{L} \geq 2v_{K(i+1)}^{L} = \sum_{a=1}^n \alpha_a K_a v_a - \frac{1}{2} \left( K_v - K^L \right)^{1/2} \left( E_{KV} - E_v \right)^{1/2} > v_{K(i+1)}^{L}, \]
\[ v_{v(i+1)}^{L} \geq v_{v(i+1)}^{L} = \frac{v_v + 1}{E_v} \left( \frac{1}{K^L} - \frac{1}{K^U} \right)^{1/2} \left( 1 - \frac{1}{E_v} \right)^{1/2} > v_{v(i+1)}^{L}, \]
\[ v_{K(i+1)}^{L} \geq v_{K(i+1)}^{L} = \frac{E_{vK} v_{U2}^{(0)} - v_{U4}^{(0)} > 0}{\alpha_a K_a v_a - \frac{1}{2} \left( K_v - K^L \right)^{1/2} \left( E_{KV} - E_v \right)^{1/2}} > v_{K(i+1)}^{L}, \]
\[ v_{K(i+1)}^{L} \geq v_{K(i+1)}^{L} = \frac{E_{vK} v_{L2}^{(0)} - v_{L4}^{(0)} > 0}{\alpha_a K_a v_a - \frac{1}{2} \left( K_v - K^L \right)^{1/2} \left( E_{KV} - E_v \right)^{1/2}} > v_{K(i+1)}^{L}, \]
\[ v_{K(i+1)}^{L} \geq v_{K(i+1)}^{L} = \max \left\{ \alpha_a K_a v_a - \frac{1}{2} \left( K_v - K^L \right)^{1/2} \left( E_{KV} - E_v \right)^{1/2} \right\} > v_{K(i+1)}^{L}. \]

(20)

The bounded monotonously increasing \( E_{K(i)}, E_{K(i)}, K_{E(i)}, v_{E(i)}, v_{E(i)}^{L}, v_{E(i)}^{L} \) and decreasing \( E_{v(i)}, K_{E(i)}^{eff} \) series will converge eventually to certain finite values \( E_{v}, v_{v}, v_{v}, v_{v}, E_{v}, K_{E} \) respectively. Besides, using unspecified bounds \( E_{K(i)}, E_{K(i)}, K_{E(i)}^{eff} \) and above inequality expressions, we have:

\[ v_{E(i+1)}^{eff} \leq v_{E(i+1)}^{eff} = \frac{E_{v} + 1}{E_v} \left( \frac{1}{K^L} - \frac{1}{K^U} \right)^{1/2} \left( 1 - \frac{1}{E_v} \right)^{1/2} > v_{E(i)}^{eff}, \]
\[ v_{K(i+1)}^{eff} \leq v_{K(i+1)}^{eff} = \max \left\{ \alpha_a K_a v_a - \frac{1}{2} \left( K_v - K^L \right)^{1/2} \left( E_{KV} - E_v \right)^{1/2} \right\} > v_{K(i)}^{eff}, \]
\[ v_{L3}^{(0)} = \sum_{a=1}^n \alpha_a K_a v_a + \frac{1}{2} \left( K_v - K^L \right)^{1/2} \left( E_{KV} - E_v \right)^{1/2}, \]

171
\[ \nu_{\text{eff}}^3 \leq V U^3 = (K_L) \nu_{V U}^3 < V U_{(0)}^3, \]
\[ \nu_{\text{eff}}^4 \leq V U^4 \leq \left( E_V K_L \right) \nu_{V U}^4 < V U_{(0)}^4, \]
\[ \nu_{\text{eff}}^4 \leq V U^4 = \left( \frac{\nu_v}{E_v} + \frac{1}{2} \frac{1}{K_L} - \frac{1}{K_L} \right)^{1/2} \left( \frac{1}{E_v} - \frac{1}{E_u} \right) < V U_{(0)}^4, \]
\[ \nu_{\text{eff}}^4 \leq V U^4 = E_V V U_{(0)}^4 < V U^4, \]
\[ \nu_{\text{eff}}^4 \leq V U^4 = K_L^* E^* V U_{(0)}^4 < V U^4_{(0)}, \]  
(21)

Finally, we get new iteration expressions:
\[
E_v \leq E^{(0)} \leq E^U, \\
E^L \leq E^L \leq E^U = E_KV, \\
\left( \frac{1}{K_L} + \frac{4\nu_v^2}{E_v} \right)^{-1} = K_L^* \leq K_L^{(0)} \leq K_L^*; \\
0 < \nu_l \leq \nu_{\text{eff}} \leq \nu_{U} = \min \{ \nu_{U1}, \nu_{U2}, \nu_{U3}, \nu_{U4} \}, \\
0 < \nu_{l} \leq \nu_{\text{eff}} \leq \nu_{K} = \min \{ \nu_{K1}, \nu_{K2}, \nu_{K3}, \nu_{K4} \}, \\
0 < \nu_{l} \leq \nu_{\text{eff}} \leq \nu_{E} = \min \{ \nu_{E1}, \nu_{E2}, \nu_{E3}, \nu_{E4} \}. 
\]

Thus, the total set of these above bounds has been constructed as directly as possible on all six longitudinal-transverse elastic coefficients for unidirectional composites. By taking the finer \( \nu_{K(-)} \nu_{V(-)} \) as the lower bound \( (K\nu^2)_{(-)} \) on \( K^{(0)} \nu(-) \left( \nu^2 \right) \) (compared to \( K^L \nu(-) \) in [14, 15]), and \( \nu_{E(-)} \nu_{V(-)} \) as the lower bound \( (\nu^2 / E_{(-)}) \) on \( \left( \nu^2 \right) \left( \nu_{(-)} \right) \) (compared to \( (\nu^2 / E_{(-)}) \) in [14, 15]), this section has improved the primary iteration bounds on \( E^{(0)}, \nu_{\text{eff}}, \) and built new direct iteration boundary expressions on \( E^{(0)}_K, K_L^{(0)} = E_K^*, K_L^{(0)} \). In comparison with primary ones in [12-15], which are intended just to the bounds on macroscopic longitudinal elastic modulus. As a result of the improved procedure, these new iteration expressions should also be better. For visualization, numerical calculation results and numerical illustrations for specific composites will be presented in the next section.

V. NUMERICAL COMPARISONS AND ILLUSTRATIONS

This section applies new iterative boundary expressions for numerical calculations for real composites with positive Poisson’s coefficient. These expressions constructed in the previous section are valid for general \( n \)-component transversely isotropic composite materials, however for simplicity we only calculate for two-component and three-component composites. Furthermore, the corresponding Voigt-Ruess and Hashin-Shtrikman boundaries have only been established for two-component composites.

Hanh Vuong et al

It has been constructed that in the special case two component unidirectional composite \( E^{\text{eff}}, \nu_{\text{eff}} \) can be expressed through \( K^{\text{eff}} \) thanks to relations Hill-Paul [1, 2]. Taking Hashin-Shtrikman bounds \( K^{U} = K^{U}_{HS}, K^{L} = K^{L}_{HS} \) and Hill’s relations, one receive Hashin-Shtrikman bounds for \( E^{\text{eff}}, \nu_{\text{eff}} \) of 2-component unidirectional composites that are statistically isotropic in the transverse plane. Alternatively, taking Hill-Paul bounds \( K^{U} = K_{V}, K^{L} = K_{L} \) one receive Hill-Paul bounds for \( E^{\text{eff}}, \nu_{\text{eff}} \) of the more general 2-component unidirectional composites that are just macroscopically isotropic in the transverse plane.

Comparing our iteration bounds with Hill-Paul one in 2-component case, we see immediately that our lower bound on \( E^{\text{eff}} \) coincides with the respective Hill-Paul one. However, as we shall see, our other bounds appear less restrictive. Because our previous reports have compared specific numbers with Voigt-Ruess and Hashin-Shtrikman, this report will not compare with them anymore.

A. Two-component case

For consistency, we take again the example considered in our previous report [15].

Consider the glass-epoxy composite (built new direct iteration boundary expressions on \( E^{(0)}_K, K_L^{(0)} = E_K^*, K_L^{(0)} \). In comparison with primary ones in [12-15], which are intended just to the bounds on macroscopic longitudinal elastic modulus. As a result of the improved procedure, these new iteration expressions should also be better. For visualization, numerical calculation results and numerical illustrations for specific composites will be presented in the next section.

\[
\begin{array}{c|c|c|c|c}
  \nu_v & E^L = E_V & E^{U} & E^{U}_{(0)} & K_{(0)}^{U} = K^{U}_{(0)} \\
\hline
  0.1 & 67.6780 & 73.4063 & 73.1561 & 73.8469 \\
  0.2 & 60.5360 & 65.8430 & 65.9994 & 66.2527 \\
  0.3 & 53.3040 & 58.3013 & 58.5206 & 58.6585 \\
  0.4 & 46.0720 & 50.8601 & 50.9728 & 51.0643 \\
  0.5 & 38.8400 & 43.2161 & 43.3997 & 43.4701 \\
  0.6 & 31.6080 & 35.6215 & 35.8128 & 35.8759 \\
  0.7 & 24.3760 & 28.0126 & 28.2094 & 28.2817 \\
  0.8 & 17.1440 & 20.3213 & 20.5015 & 20.6875 \\
  0.9 & 9.9120 & 12.5137 & 12.7009 & 13.0933 \\
\end{array}
\]

TABLE I. NUMERICAL RESULTS AND COMPARISONS OF YOUNG’S MODULUS FOR GLASS-EPoxy COMPOSITE
Upper and lower iteration boundaries on longitudinal-transverse elastic modulus of transversely isotropic unidirectional composites

TABLE II. NUMERICAL RESULTS AND COMPARISONS OF POISSON’S RATIO FOR GLASS-EPOXY COMPOSITE

<table>
<thead>
<tr>
<th>$\nu_e$</th>
<th>$\nu_L(0)=\nu_{\nu L}L(0)$</th>
<th>$\nu_T(0)=\nu_{\nu T}T(0)$</th>
<th>$\nu_U(0)=\nu_{\nu U}U(0)$</th>
<th>$\nu_{UU}(0)$</th>
<th>$\nu_{UU}(0)$</th>
<th>$\nu_{UU}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0726</td>
<td>0.0802</td>
<td>0.0984</td>
<td>0.1914</td>
<td>0.2174</td>
<td>0.2201</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0553</td>
<td>0.0591</td>
<td>0.0701</td>
<td>0.1995</td>
<td>0.2639</td>
<td>0.2715</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0476</td>
<td>0.0502</td>
<td>0.0895</td>
<td>0.2630</td>
<td>0.2987</td>
<td>0.3019</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0458</td>
<td>0.0459</td>
<td>0.0967</td>
<td>0.2987</td>
<td>0.3316</td>
<td>0.3408</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0424</td>
<td>0.0443</td>
<td>0.0838</td>
<td>0.3348</td>
<td>0.3593</td>
<td>0.3617</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0436</td>
<td>0.0456</td>
<td>0.0607</td>
<td>0.3609</td>
<td>0.3750</td>
<td>0.3802</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0497</td>
<td>0.0525</td>
<td>0.0701</td>
<td>0.3714</td>
<td>0.3831</td>
<td>0.3901</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0819</td>
<td>0.0900</td>
<td>0.1054</td>
<td>0.3809</td>
<td>0.3931</td>
<td>0.3997</td>
</tr>
<tr>
<td>0.9</td>
<td>0.1195</td>
<td>0.1390</td>
<td>0.2048</td>
<td>0.3864</td>
<td>0.3957</td>
<td>0.4015</td>
</tr>
</tbody>
</table>

Comments on Table I, II:
+ The initial values $E^L = E_L$, $E^T_{(0)} = E_{KT}$, $\nu^L(0) = \nu_{\nu L}L(0)$, $\nu^T_{(0)} = \nu_{\nu T}T(0)$ remain the same as the primary respective ones.
+ The new iteration bounds $E^U$, $\nu^T$, $\nu^U$ improve upon the primary ones $E^T$, $\nu^T$, $\nu^T$, as expected.

Here, we take Hashin-Strikman bounds $K^U = K^U_{HS}$, $K^L = K^L_{HS}$ in the intermediate steps as in [12-15], therefore these bounds are valid for those unidirectional composites that are statistically isotropic in the transverse plane.

- Illustrations of $E^{eff}$, $\nu^{eff}$, $K^{eff}$ for glass-epoxy composite

Graphical illustrations to compare the iteration (convergence) values with the corresponding initial values of all longitudinal-transverse elastic modulus modules are presented in figures 1-6 below.

Fig. 1. Illustrations of Young’s modulus for glass-epoxy composite

Fig. 2. Illustrations of Poisson’s ratio for glass-epoxy composite

Fig. 3. Illustrations of $E^{eff}$ for glass-epoxy composite

Fig. 4. Illustrations of $K^{eff}$ for glass-epoxy composite

Fig. 5. Illustrations of $\nu^{eff}$ for glass-epoxy composite

Fig. 6. Illustrations of $\nu^{eff}$ for glass-epoxy composite
Comments on Figs. 1-6:
+ The lower and upper bounds on $E^{\text{eff}}, \nu^{\text{eff}}, E_K^{\text{eff}}, K^{\text{eff}}$, $\nu_K^{\text{eff}}, \nu_{\text{eff}}$ appear very good and very good respectively.
+ The convergence values $E, \nu, E_K, K, \nu_K$ have improved over the corresponding initial values $E_i(0), \nu_i(0), E_K(0), K(0), \nu_K(0)$, whereas these initial values have made the main part of the improvement.
+ When the ratio of glass and epoxy changes, the iteration values converge quickly, the boundary series almost converge when $i = 8$ (with convergence error $\epsilon \leq 10^{-4}$).

B. Three-component case

Similarly, consider the aluminum-copper-steel composite (considered in [15]) with the elastic coefficients (unit GPa) and the volume ratio of the corresponding components as:
- Aluminum: $E_a = 68.34, K_a = 71.19, \mu_a = 25.5, \nu_a = 0.34$,
- Copper: $E_c = 118.9, K_c = 116.57, \mu_c = 44.7, \nu_c = 0.33$,
- Steel: $E_s = 206.4, K_s = 172.03, \mu_s = 79.3, \nu_s = 0.3$.

Volume ratio: $\nu_a = 0.1 \rightarrow 0.6, \nu_c = 0.3$.

Using Matlab, calculate the new iteration bounds $E^{d}, E^{u}, \nu^{d}, \nu^{u}$, and compare to primary ones $E^{d}, \nu^{d}$, we get results of $E^{\text{eff}}$ and $\nu^{\text{eff}}$ presented in Table III and Table IV, respectively (with convergence error $\epsilon \leq 10^{-4}$).

**TABLE III. NUMERICAL RESULTS AND COMPARISONS OF YOUNG’S MODULUS FOR ALUMINUM-COPPER-STEEL COMPOSITE**

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$E^{d} = E_0$</th>
<th>$E^{u}$</th>
<th>$E^{d\nu}$</th>
<th>$E^{d\mu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>97.3109</td>
<td>104.651</td>
<td>105.904</td>
<td>138.4914</td>
</tr>
<tr>
<td>0.2</td>
<td>111.120</td>
<td>122.003</td>
<td>122.539</td>
<td>155.1967</td>
</tr>
<tr>
<td>0.3</td>
<td>124.926</td>
<td>137.518</td>
<td>138.105</td>
<td>171.9059</td>
</tr>
<tr>
<td>0.4</td>
<td>138.732</td>
<td>152.538</td>
<td>151.491</td>
<td>186.6132</td>
</tr>
<tr>
<td>0.5</td>
<td>152.538</td>
<td>164.369</td>
<td>165.556</td>
<td>205.3204</td>
</tr>
<tr>
<td>0.6</td>
<td>166.344</td>
<td>175.281</td>
<td>171.905</td>
<td>222.0277</td>
</tr>
</tbody>
</table>

**TABLE IV. NUMERICAL RESULTS AND COMPARISONS OF POISSON’S RATIO FOR ALUMINUM-COPPER-STEEL COMPOSITE**

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$\nu^{d\nu} = \nu_0(0)$</th>
<th>$\nu^{d\mu}$</th>
<th>$\nu^{u\nu}$</th>
<th>$\nu^{u\mu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.2531</td>
<td>0.3046</td>
<td>0.3102</td>
<td>0.3401</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2406</td>
<td>0.2932</td>
<td>0.2996</td>
<td>0.3353</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2341</td>
<td>0.2862</td>
<td>0.2905</td>
<td>0.3306</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2321</td>
<td>0.2831</td>
<td>0.2897</td>
<td>0.3257</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2345</td>
<td>0.2838</td>
<td>0.2906</td>
<td>0.3214</td>
</tr>
<tr>
<td>0.6</td>
<td>0.2425</td>
<td>0.2847</td>
<td>0.2918</td>
<td>0.3102</td>
</tr>
</tbody>
</table>

Comments on Table III, IV:

Similar to the 2-component case above, the new iteration values on two macroscopic longitudinal elastic modulus have improved compared to the primary ones.

Illustrations of six longitudinal-transverse elastic modulus for aluminum-copper-steel composite are also similar to the glass-epoxy case mentioned above.

**CONCLUSION**

Starting from the principles of minimum energy, with combination inequalities, initial and iteration bounds on all six effective longitudinal-transverse elastic coefficients of general $n$-component unidirectional transversely isotropic composite has been established.

By taking the better undefined available estimates adding new boundary sequences in iterative process, this report has improved the primary iteration ones on two macroscopic longitudinal elastic modulus. The validity of these new expressions has been proven by numerical comparison results for 2- and 3-component composites.

Iteration and optimization techniques have been applied to obtain best possible estimates for six elastic modulus from the mixed longitudinal-transverse stress-strain modes. Illustrations of these estimates are also considered in real 2-component composite case.

**ACKNOWLEDGMENT**

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**REFERENCES**